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Proving the accuracy of invariant operator approach for quantum solutions of time-dependent coupled oscillators including relevant criticisms and perspectives

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ABSTRACT

Choi recently published a research paper on the subject of quantizing general time-dependent coupled oscillators (arXiv:2210.07551v1 [quant-ph] (2022)) using the invariant operator approach. In the present research, we provide a direct method for proving the correctness of such an invariant approach, together with relevant perspectives and criticisms. The difference of our method from Choi's one in this context is that, while Choi managed the quantum system with the aid of the unitary transformation technique, we do not use such an auxiliary tool in our proof of accuracy. This demonstration may provide a definite mathematical testimony for the usefulness of adopting invariant operator approach in a straightforward way. Our proof is crucial in unfolding quantum information theory including opto-electromechanics via coupled devices which operate with electro-mechanical oscillations. We also discuss the consequences of Choi's work and Zerimeche *et al.*'s works (Mod. Phys. Lett. B 37, 2250222 (2023) and others) which share many concepts with each other, including the relation between them.

Keywords: Coupled oscillators, Quantum dynamics, Unitary transformation, Quantum solutions, Invariant operator

1. INTRODUCTION

Coupled systems are ubiquitous in scientific disciplines, such as physics, chemistry, and biology. Some of them are quantum devices, molecular clusters, neural networks, and nano-optomechanical resonators. Mutual interactions between their constituents become high as the coupling strength increases. Synchronization, resonance, and nonlinearity of systems under such interactions, as well as their stability, are of

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interest and widely analyzed so far. In connection with this, entanglement and superposition are novel quantum phenomena appearing in coupled systems (Alet et al., 2021; Sakkaf and Khawaja, 2020). While mathematical frameworks of these two physical effects are independent, both of them are non-intuitive. Nevertheless, mixed entangled quantum states together with coherent superposition states can be considerably applied to quantum information processing. In particular, entangled states are at the heart of developing next-generation quantum technologies.

Coupled oscillators are used in modeling a wide range of mutually connected systems, such as complex mechanical, electrical, and photonic devices that exhibit intricate oscillatory behavior. Furthermore, coupled oscillators are applied to diverse modern technologies, as well as in traditional analyses of crystal lattice vibrations (Wang et al., 2018; Schettino, 2018) and motion of diatomic molecules (Čáp et al., 2021). Some of emergent technologies utilizing coupled oscillators are quantum computing and cryptography (Dudas et al., 2023; Csaba and Porod, 2020; Choi and Ju, 2019), locomotion synchronization in bio-inspired robotics (Dutta et al., 2019), and electromagnetically induced transparency (Litvak and Tokman, 2002; Alzar et al., 2021). Complicated dynamical behaviors of these systems can be studied based on their exact or approximated quantum solutions.

In many actual cases, modeling of systems by coupled oscillatory motions is non-ideal and, hence, they are managed approximately because of the mathematical difficulty in deriving their accurate solutions (Abdalla et al., 1998; El-Orany et al., 2004; Augier et al., 2022; Choi, 2021). In particular, if the parameters vary over time, the motion of coupled oscillators may be very complicated (Csaba and Porod, 2020). Recently, Choi showed how to manage coupled time-dependent oscillators quantum-mechanically via the invariant operator approach (Choi, 2022a). The strategy used by Choi in that work is the use of the unitary transformations in unfolding his new formula of the invariant operator for the coupled systems. For time-varying systems, the eigenstates of the invariant operator constitute wave functions which are Schrödinger solutions.

In this research, we prove that the solutions obtained based on such invariant operator approach are exact. Though Choi managed the system with the aid of the unitary transformation technique, we use a direct way without relying on that technique in demonstrating the exactness of the operator approach. Our demonstration may provide a complete mathematical background for quantum dynamics of the coupled systems, enriching our understanding of the quantum characteristics of time-dependent oscillatory systems. For instance, it may be a basis for clarifying the mechanism of entanglement between time-varying coupled oscillators, as well as making the analysis of superposition states possible. Such a mechanism is important in realizing quantum information protocols for state-of-the-art computing, communication, and cryptography in the future.

We organize this work as follows. In Sec. 2, we demonstrate the correctness of the quantum-dynamical invariant formulation for coupled time-dependent oscillators. Eventually, we prove straightforwardly that the quantum solutions obtained using dynamical invariant are exact. We discuss another research work (Zerimeche et al., 2023) in connection with time-dependent coupled oscillators including its incorrectness and the relations of its content with Choi's one in Sec. 3. The indepth comparison of Ref. (Zerimeche et al., 2023) with Choi's one (Choi, 2022a) is also carried out in Sec. 4. Some perspectives and criticisms of other works of the authors of Ref. (Zerimeche et al., 2023) are also addressed in the same section. Concluding remarks are given in the last section.

2. DIRECT METHOD FOR DEMONSTRATING THE CORRECTNESS OF THE INVARIANT OPERATOR APPROACH

If the oscillators are non-coupled, the treatment of the systems may be relatively simple and the solutions are well-known. However, for the case of coupled oscillators, their rigorous treatment may be not easy because the motions are complex. We can also think even the case when the parameters vary over time. For such time-varying coupled oscillators, Choi used invariant operator approach in order to derive quantum solutions.

In this section, we directly demonstrate that the quantum solutions obtained based on invariant operator approach are mathematically correct, i.e., we verify that they can be derivable without any approximation or perturbation manipulations, as well as unitary transformation techniques. To this end, we show the correctness of the invariant operator by direct differentiation of it with respect to time as a preliminary step. And then we prove that the eigenvalue equation for such an invariant operator gives exact quantum solutions in a straightforward way. We also see the significance of such solutions in unfolding quantum dynamics of the systems.

2.1. Proving the accuracy of the invariant operator

The Hamiltonian of the coupled harmonic oscillators with time-varying parameters, which we consider, is given by

$$\hat{H}(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\frac{\hat{p}_i^2}{m_i(t)} + b_i(t)(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i) + m_i(t)\omega_i^2(t)\hat{x}_i^2\right] + d(t)\hat{x}_1\hat{x}_2, \tag{1}$$

where $\hat{p}_i = -i\hbar\partial/\partial x_i$, while $m_i(t)$ are effective masses, $\omega_i(t)$ are angular frequencies, $d(t)$ is a coupling strength, and $b_i(t)$ are parameters that are real. We note that all the parameters in this equation are dependent on time. If the Hamiltonian of the system varies in time like in this case, quantum solutions are not represented by the eigenstates of the Hamiltonian, but by the eigenstates of an invariant operator (Lewis, 1967). According to this, Choi introduced a quadratic invariant operator for $\hat{H}(t)$, which is represented as (Choi, 2022a)

$$\hat{I}(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\alpha_i(t)\hat{p}_i^2 + \beta_i(t)(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i) + \gamma_i(t)\hat{x}_i^2\right] + \delta(t)\hat{x}_1\hat{x}_2. \tag{2}$$

Here, the coefficients of the invariant are given by

$$\alpha_i(t) = \alpha_{0,i}\rho_i^2(t), \tag{3}$$

$$\beta_i(t) = \alpha_{0,i}m_i(t)[b_i(t)\rho_i^2(t) - \rho_i(t)\dot{\rho}_i(t)], \tag{4}$$

$$\gamma_i(t) = \alpha_{0,i}\left[\frac{\Omega_i^2}{4\rho_i^2(t)} + m_i^2(t)\left(b_i^2(t)\rho_i^2(t) - 2b_i(t)\rho_i(t)\dot{\rho}_i(t) + \dot{\rho}_i^2(t)\right)\right], \tag{5}$$

$$\delta(t) = F(t)d(t), \tag{6}$$

where $\alpha_{0,i}$ are real constants, $F(t)$ is of the form

$$F(t) = \alpha_1(t)m_1(t) \;\; (= \alpha_2(t)m_2(t)), \tag{7}$$

and $\rho_i(t)$ are the solutions of the following equations:

$$\ddot{\rho}_i(t) = -\frac{\dot{m}_i(t)}{m_i(t)}\dot{\rho}_i(t) - \tilde{\omega}_i^2(t)\rho_i(t) + \frac{\Omega_i^2}{4m_i^2(t)\rho_i^3(t)}, \tag{8}$$

with $\tilde{\omega}_i^2(t) = \omega_i^2 - b_i^2 - \dot{b}_i - b_i\dot{m}_i/m_i$ and the real constants Ω_i.

We now see whether $\hat{I}(t)$ is correct or not as an invariant operator by examining its time derivative. That is, if $d\hat{I}(t)/dt = 0$, $\hat{I}(t)$ is the exact invariant operator. The detailed result of the evaluation for this derivative is

$$\frac{d\hat{I}(t)}{dt} = \sum_{i=1}^{2}[f_i(t)\hat{x}_i^2 + g_i(t)(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i)] + h_1(t)\hat{p}_1\hat{x}_2 + h_2(t)\hat{p}_2\hat{x}_1 + z(t)\hat{x}_1\hat{x}_2, \tag{9}$$

where

$$\begin{aligned} f_i(t) = \alpha_{0,i}[b_i(t)\rho_i(t) - \dot{\rho}_i(t)]\{\Omega_i^2/[4\rho_i^3(t)] + b_i^2(t)m_i^2(t)\rho_i(t) \\ + b_i(t)m_i(t)\dot{m}_i(t)\rho_i(t) - m_i(t)\dot{m}_i(t)\dot{\rho}_i(t) \\ - m_i^2(t)\{\rho_i(t)[\omega_i^2(t) - \dot{b}_i(t)] + \ddot{\rho}_i(t)\}\}, \end{aligned} \tag{10}$$

$$\begin{aligned} g_i(t) = \frac{\alpha_{0,i}}{2}\{\Omega_i^2/[4m_i(t)\rho_i^2(t)] + b_i^2(t)m_i(t)\rho_i^2(t) + b_i(t)\dot{m}_i(t)\rho_i^2(t) \\ - \dot{m}_i(t)\rho_i(t)\dot{\rho}_i(t) - m_i(t)\rho_i(t)\{\rho_i(t)[\omega_i^2(t) - \dot{b}_i(t)] + \ddot{\rho}_i(t)\}\}, \end{aligned} \tag{11}$$

$$h_i(t) = d(t)[F(t)/m_i(t) - \alpha_{0,i}\rho_i^2(t)], \tag{12}$$

$$\begin{aligned} z(t) = d(t)\sum_{i=1}^{2}\{b_i(t)[F(t) - \alpha_{0,i}m_i(t)\rho_i^2(t)] + \alpha_{0,i}m_i(t)\rho_i(t)\dot{\rho}_i(t)\} \\ + \dot{F}(t)d(t) + F(t)\dot{d}(t), \end{aligned} \tag{13}$$

whereas $i = 1, 2$.

If we use Eq. (8) for $\ddot{\rho}_i(t)$, $f_i(t)$ and $g_i(t)$ become zero. An additional use of the relation

$$F(t) = \alpha_i(t)m_i(t) = \alpha_{0,i}\rho_i^2(t)m_i(t), \tag{14}$$

leads $h_i(t) = 0$. Lastly, if we use Eq. (14) for $F(t)$ and the following constraint for $\dot{d}(t)$

$$\dot{d}(t) = -G(t)d(t), \tag{15}$$

where

$$G(t) = \frac{\dot{m}_1(t)}{m_1(t)} + \frac{3\dot{\rho}_1(t)}{\rho_1(t)} + \frac{\dot{\rho}_2(t)}{\rho_2(t)}, \tag{16}$$

we acquire $z(t) = 0$. The relation in Eq. (15) has been introduced by Choi (see Eq. 19 of Ref. (Choi, 2022a)). We now readily see that $d\hat{I}(t)/dt = 0$ from the results shown up until now. Therefore, $\hat{I}(t)$ is a time constant, implying that it is exact as a quadratic invariant operator.

2.2. Direct method for demonstrating quantum solutions

Choi's eigenstates are given in Eq. 45 of Ref. (Choi, 2022a). These states are obtained using the invariant operator given in the above subsection and they are represented as

$$\langle x_1, x_2 \mid \varphi_{n_1,n_2} \rangle = \prod_{i=1}^{2} \sqrt[4]{\frac{\varpi_{0,i}}{\pi\hbar\alpha_i(t)}} \frac{1}{\sqrt{2^{n_i} n_i!}} H_{n_i}\left(\sqrt{\frac{\varpi_{0,i}}{\hbar}} X_i\right) \exp\left[-\frac{1}{2\hbar}\left(\varpi_{0,i} X_i^2 + i\frac{\beta_i(t)}{\alpha_i(t)} x_i^2\right)\right], \tag{17}$$

where H_{n_i} are Hermite polynomials, and

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\alpha_1(t)}}\cos\varphi & \frac{1}{\sqrt{\alpha_2(t)}}\sin\varphi \\ -\frac{1}{\sqrt{\alpha_1(t)}}\sin\varphi & \frac{1}{\sqrt{\alpha_2(t)}}\cos\varphi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{18}$$

Here we prove that the invariant operator approach gives exact quantum result, by examining the eigenstates of Choi, $\langle x_1, x_2 \mid \varphi_{n_1,n_2} \rangle$, in a direct way. If we consider the eigenvalue equations of the invariant operator in the configuration spaces, which are

$$K_{n_1,n_2} \equiv \left[I\left(x_1, x_2, \frac{\hbar}{i}\frac{\partial}{\partial x_1}, \frac{\hbar}{i}\frac{\partial}{\partial x_2}, t\right) - \lambda_{n_1,n_2}\right] \langle x_1, x_2 \mid \varphi_{n_1,n_2} \rangle, \tag{19}$$

where (see Eq. 41 in Ref. (Choi, 2022a))

$$\lambda_{n_1,n_2} = \sum_{i=1}^{2} \hbar\varpi_{0,i}\left(n_i + \frac{1}{2}\right), \tag{20}$$

such a proof can be done by showing that $K_{n_1,n_2} = 0$.

If we see the formula of I from Eq. (2), the evaluation of $I\langle x_1, x_2 \mid \varphi_{n_1,n_2} \rangle$ requires the first and second order derivatives of $\langle x_1, x_2 \mid \varphi_{n_1,n_2} \rangle$ with respect to x_i, while $\langle x_1, x_2 \mid \varphi_{n_1,n_2} \rangle$ involve Hermite polynomials. This evaluation can be done by using the relation

$$\frac{dH_n(y)}{dy} = 2nH_{n-1}(y). \tag{21}$$

The detailed expression of K_{n_1,n_2} derived by means of this relation is of the form

$$K_{n_1,n_2} = \left[\hbar\sum_{i=1}^{2} \varpi_{0,i}\left(2n_i \frac{H_{n_i-2}(y_i)(1-n_i) + y_i H_{n_i-1}(y_i)}{H_{n_i}(y_i)} - n_i\right) + k_0 x_1 x_2 + k_1 x_1^2 + k_2 x_2^2\right] \langle x_1, x_2 \mid \varphi_{n_1,n_2} \rangle, \tag{22}$$

where

$$y_1 = \sqrt{\frac{\varpi_{0,1}}{\hbar}}\left(\frac{\cos\varphi}{\sqrt{\alpha_1(t)}} x_1 + \frac{\sin\varphi}{\sqrt{\alpha_2(t)}} x_2\right), \tag{23}$$

$$y_2 = \sqrt{\frac{\varpi_{0,2}}{\hbar}}\left(-\frac{\sin\varphi}{\sqrt{\alpha_1(t)}}x_1 + \frac{\cos\varphi}{\sqrt{\alpha_2(t)}}x_2\right), \tag{24}$$

$$k_0 = \delta - \frac{\varpi_{0,1}^2 - \varpi_{0,2}^2}{2\sqrt{\alpha_1(t)\alpha_2(t)}}\sin(2\varphi), \tag{25}$$

$$k_1 = \frac{1}{4\alpha_1(t)}\{2[\alpha_1(t)\gamma_1(t) - \beta_1^2(t)] - \varpi_{0,1}^2 - \varpi_{0,2}^2 - (\varpi_{0,1}^2 - \varpi_{0,2}^2)\cos(2\varphi)\}, \tag{26}$$

$$k_2 = \frac{1}{4\alpha_2(t)}\{2[\alpha_2(t)\gamma_2(t) - \beta_2^2(t)] - \varpi_{0,1}^2 - \varpi_{0,2}^2 + (\varpi_{0,1}^2 - \varpi_{0,2}^2)\cos(2\varphi)\}. \tag{27}$$

As a next step, we can simplify K_{n_1,n_2} by eliminating the terms involving $H_{n_j-2}(y_j)$ from their formulae given in Eq. (22) through the replacement (Bateman, 1953)

$$H_{l_j-1}(y_j) \quad \to \quad \frac{1}{2l_j}[2y_jH_{l_j}(y_j) - H_{l_j+1}(y_j)], \tag{28}$$

under the choice of $l_j = n_j - 1$. Then all terms involving n_j in Eq. (22) are canceled out, leading to

$$K_{n_1,n_2} = [k_0x_1x_2 + k_1x_1^2 + k_2x_2^2]\langle x_1, x_2 \mid \varphi_{n_1,n_2}\rangle. \tag{29}$$

Now by inserting Eqs. (25)-(27) into the above equation, and then by replacing (see Eqs. 26, 30, and 31 in Ref. (Choi, 2022a))

$$\alpha_j(t)\gamma_j(t) - \beta_j^2(t) \;\to\; \omega_{0,j}^2, \tag{30}$$

$$\varpi_{0,1}^2 \;\to\; \omega_{0,1}^2\cos^2\varphi + \omega_{0,2}^2\sin^2\varphi + \delta(t)\sqrt{\alpha_1(t)\alpha_2(t)}\sin(2\varphi), \tag{31}$$

$$\varpi_{0,2}^2 \;\to\; \omega_{0,1}^2\sin^2\varphi + \omega_{0,2}^2\cos^2\varphi - \delta(t)\sqrt{\alpha_1(t)\alpha_2(t)}\sin(2\varphi), \tag{32}$$

we arrive

$$\frac{K_{n_1,n_2}}{\langle x_1, x_2 \mid \varphi_{n_1,n_2}\rangle} = \frac{\bar{\delta}}{M}\left[\frac{\cos(2\varphi)}{\sqrt{\alpha_1(t)\alpha_2(t)}}x_1x_2 - \frac{\sin(2\varphi)}{2}\left(\frac{x_1^2}{\alpha_1(t)} - \frac{x_2^2}{\alpha_2(t)}\right)\right], \tag{33}$$

where

$$\bar{\delta} = M\left(\delta(t)\sqrt{\alpha_1(t)\alpha_2(t)}\cos(2\varphi) - \frac{1}{2}(\omega_{0,1}^2 - \omega_{0,2}^2)\sin(2\varphi)\right). \tag{34}$$

By choosing φ as (Choi, 2022a)

$$\varphi = \frac{1}{2}\mathrm{atan}\left((\omega_{0,1}^2 - \omega_{0,2}^2)/2, \delta(t)\sqrt{\alpha_1(t)\alpha_2(t)}\right), \tag{35}$$

where $\vartheta \equiv \mathrm{atan}(z_1, z_2)$ is two-variables arctangent function of $\tan\vartheta = z_2/z_1$, we readily see that $\bar{\delta}$ is being zero. Hence $K_{n_1,n_2} = 0$, confirming that the solutions, Eq. (17), and the corresponding eigenvalues, Eq. (20), are exact.

2.3. Significance of the invariant operator approach

From previous evaluations, we have proved, in a direct way, that the quantum solutions for coupled time-dependent oscillators relevant to invariant operator approach are exact. That is, no approximations or perturbation techniques were used in this demonstration. The exactness of such solutions may play the role of a central key in the mathematical treatment of quantum systems that are coupled one another. Such solutions enable us to understand entanglement between coupled devices, making it possible to find a strategy for its optimal control. Quantum entanglement is notable especially in nano-dimensions where quantum effects are prominent (Villar et al., 2020; Li and Kröger, 2012). Based on our outcome, we can analyze quantum characteristics of generally coupled oscillatory systems represented by a time-varying Hamiltonian in an exact way: for instance, it is possible to analyze intrinsic quantum

uncertainties, quadrature squeezing, quantum measurements, and wave collapses by decoherence, as well as entanglement that is relatively well-known.

3. PERSPECTIVE OF ANOTHER SIMILAR WORK

Zerimeche *et al*. recently published a research paper entitled "On the quantum dynamics of a general time-dependent coupled oscillator" (see Ref. (Zerimeche et al., 2023)). They treated time-dependent coupled oscillators of which Hamiltonian is given by

$$\hat{H}_Z(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\frac{\hat{p}_i^2}{m_i(t)} + c_i(t)\hat{x}_i^2\right] + \frac{1}{2}c_3(t)\hat{x}_1\hat{x}_2, \tag{36}$$

where $c_1(t)$, $c_2(t)$, and $c_3(t)$ are time-dependent parameters that are real. Some notations in the above expression have been changed from those of Ref. (Zerimeche et al., 2023) for the consistency of our representation without loss of generality (this convention will also be applied for expressing their other notations if necessary). Zerimeche *et al*. represented the theory of obtaining quantum solutions of the system described by this Hamiltonian in sections 2.1 and 2.2 in Ref. (Zerimeche et al., 2023). They tried to establish a dynamical invariant of the system in section 2.1 and to obtain quantum mechanical solutions based on unitary transformations of the invariant operator in section 2.2.

Zerimeche *et al*. explained that they adopted the method of Macedo and Guedes (Macedo and Guedes, 2012) in the development of the relevant theory. However, we strongly doubt that, through Ref. (Zerimeche et al., 2023), they have adopted Choi's theory of Ref. (Choi, 2022a) instead of that method. The relation between the Hamiltonian of the present work (which is the same as the Choi's one) and the Zerimeche *et al*.'s one (Eq. (36)) is as follows

$$\hat{H}(t) = \hat{H}_Z(t) + \frac{1}{2}\sum_{i=1}^{2} b_i(t)(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i), \tag{37}$$

whereas $\hat{H}_Z(t)$ in this representation is the same as Eq. (36) (but with slightly different notations). Based on this equation, we confirm that the system chosen by Zerimeche *et al*. is a particular case of Ref. (Choi, 2022a), where $b_i(t) = 0$. Because they did not consider $b_i(t)$ throughout Ref. (Zerimeche et al., 2023), we will consider only the case where $b_i(t) = 0$ in the subsequent arguments regarding their paper.

The invariant operator that Zerimeche *et al*. adopted in relation with the Hamiltonian, Eq. (36), is of the form (see Eq. 2 of Ref. (Zerimeche et al., 2023))

$$\hat{I}_Z(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\alpha_i(t)\hat{p}_i^2 + \beta_i(t)(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i) + \gamma_i(t)\hat{x}_i^2\right] + \frac{1}{2}\eta(t)\hat{x}_1\hat{x}_2, \tag{38}$$

where $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$, and $\eta(t)$ are time-dependent coefficients. In order to derive quantum solutions of their system, they considered unitary transformations of this operator like in the case of the Choi's paper.

During such a process in Ref. (Zerimeche et al., 2023), Zerimeche *et al*. cited Choi's work, Ref. (Choi, 2022a), one time. *However, the problem is that their use of the content of Ref. (Choi, 2022a) is too large beyond the generally permissible limit from the public view points*. The whole part of section 2.2 in Ref. (Zerimeche et al., 2023) is just Choi's theory (Choi, 2022a) that is not published in a regular journal yet. If we regard that the central part of their theory (but, actually, not their theory) is section 2.2, the problem of Ref. (Zerimeche et al., 2023) may not be trivial.

3.1. About the substantial overlapping part

The problematic part in Ref. (Zerimeche et al., 2023), which overlaps with the content of Ref. (Choi, 2022a), can be clearly seen from the two explanations in this section. Additional remarks in relation with this will also be provided separately.

3.1.1. Explanation 1

Choi used his unique two-step unitary transformation method in order to solve quantum problem of the time-dependent coupled oscillators. In the first step of transformation, the terms in the square bracket of Eq. (2) were converted to those of the simple harmonic oscillator (SHO). And, in the second step of transformation, he rotated the invariant operator for completion of its diagonalization.

Zerimeche *et al*. also adopted two-step unitary transformation technique in Ref. (Zerimeche et al., 2023), which was devised by Choi in Ref. (Choi, 2022a). They performed the unitary transformation of the invariant operator using the following unitary operators in order:

i) The unitary operator used in the first step of transformation is (Zerimeche *et al*.)

$$\hat{U}_{Z,1} = \hat{U}^{A}_{Z,1}\hat{U}^{B}_{Z,1}, \quad \text{(Eq. 20 in Ref. (Zerimeche et al., 2023))}$$

where

$$\hat{U}^{A}_{Z,1} = \prod_{i=1}^{2} \exp\left(\frac{i}{2\hbar}(\hat{p}_i\hat{x}_i + \hat{x}_i\hat{p}_i)\ln\sqrt{\alpha_i}\right), \quad \text{(Eq. 21 in Ref. (Zerimeche et al., 2023))}$$

$$\hat{U}^{B}_{Z,1} = \sum_{i=1}^{2} \exp\left(\frac{i}{2\hbar}\frac{\beta_i}{\alpha_i}\hat{x}_i^2\right). \quad \text{(Eq. 22 in Ref. (Zerimeche et al., 2023))}$$

ii) The unitary operator used in the second step of transformation is (Zerimeche *et al.*)

$$\hat{U}_{Z,2} = \exp\left(\frac{i\theta}{2\hbar}(\hat{p}_2\hat{x}_1 - \hat{p}_1\hat{x}_2)\right), \quad \text{(Eq. 29 in Ref. (Zerimeche et al., 2023))}$$

where θ is a phase. Let us compare the above operators with the ones in Ref. (Choi, 2022a) in turn, which are:
i) The unitary operator used in the first step of transformation is (Choi)

$$\hat{U}_1 = \hat{U}_1^{A}\hat{U}_1^{B}, \quad \text{(Eq. 21 in Ref. (Choi, 2022a))}$$

where

$$\hat{U}_1^{A} = \prod_{i=1}^{2} \exp\left(\frac{i}{2\hbar}(\hat{p}_i\hat{x}_i + \hat{x}_i\hat{p}_i)\ln\sqrt{\frac{1}{M\alpha_i}}\right), \quad \text{(Eq. 22 in Ref. (Choi, 2022a))}$$

$$\hat{U}_1^{B} = \exp\left(-\frac{i}{2\hbar}\sum_{i=1}^{2} M\beta_i\hat{x}_i^2\right). \quad \text{(Eq. 23 in Ref. (Choi, 2022a))}$$

ii) The unitary operator used in the second step of transformation is (Choi)

$$\hat{U}_2 = \exp\left(-\frac{i\varphi}{\hbar}(\hat{p}_2\hat{x}_1 - \hat{p}_1\hat{x}_2)\right). \quad \text{(Eq. 28 in Ref. (Choi, 2022a))}$$

In the expression of $\hat{U}_{Z,2}$, Zerimeche *et al*. used $\theta = 2\varphi$ where φ is a rotation angle introduced in Choi's paper. The frame of their chosen unitary operators is basically the same as Choi's one, except for the choice of $M = 1$ and a little different arrangement of α_i in $\hat{U}^{A}_{Z,1}$ and $\hat{U}^{B}_{Z,1}$. That is, $\hat{U}^{A}_{Z,1}$ (and $\hat{U}_1^{A}$ in Choi's work) is expressed in terms of $\hat{p}_i\hat{x}_i + \hat{x}_i\hat{p}_i$, $\hat{U}^{B}_{Z,1}$ (and $\hat{U}_1^{B}$ in Choi's work) is expressed in terms of $\hat{x}_i^2$, and $\hat{U}_{Z,2}$ (and $\hat{U}_2$ in Choi's work) is expressed in terms of $\hat{p}_2\hat{x}_1 - \hat{p}_1\hat{x}_2$. This implies that they have mimicked Choi's theory given in Ref. (Choi, 2022a).

In fact, for an operator $\hat{O}$, Zerimeche *et al*. considered the transformation of the type $\hat{O}' = \hat{U}_{Z,j}\hat{O}\hat{U}^{+}_{Z,j}$ ($j = 1, 2$), while Choi considered the transformation $\hat{O}' = \hat{U}_j^{-1}\hat{O}\hat{U}_j$ ($\equiv \hat{U}_j^{+}\hat{O}\hat{U}_j$) in Ref. (Choi, 2022a). Because they defined unitary operators inversely compared to Choi's ones, let us introduce $\hat{\mathcal{U}}_1^{A} = \hat{U}_1^{A+}$, $\hat{\mathcal{U}}_1^{B} = \hat{U}_1^{B+}$, and $\hat{\mathcal{U}}_2 = \hat{U}_2^{+}$ for Choi's operators, such that

$$\hat{\mathcal{U}}_1^{A} = \prod_{i=1}^{2} \exp\left(\frac{i}{2\hbar}(\hat{p}_i\hat{x}_i + \hat{x}_i\hat{p}_i)\ln\sqrt{M\alpha_i}\right), \tag{39}$$

$$\hat{\mathcal{U}}_1^{B} = \exp\left(\frac{i}{2\hbar}\sum_{i=1}^{2} M\beta_i\hat{x}_i^2\right), \tag{40}$$

$$\hat{\mathcal{U}}_2 = \exp\left(\frac{i\varphi}{\hbar}(\hat{p}_2\hat{x}_1 - \hat{p}_1\hat{x}_2)\right). \tag{41}$$

If we compare these operators, instead of $\hat{U}_1^{A}$, $\hat{U}_1^{B}$, and $\hat{U}_2$, with their operators $\hat{U}^{A}_{Z,1}$, $\hat{U}^{B}_{Z,1}$, and $\hat{U}_{Z,2}$ in turn, we can clearly see that Zerimeche *et al*. tried to use Choi's work, Ref. (Choi, 2022a).

3.1.2. Explanation 2

In order to show that their transformed invariant operator (Eq. 33 in Ref. (Zerimeche et al., 2023)) is independent of time, Zerimeche *et al*. followed the same procedure given in Ref. (Choi, 2022a). This can be seen by the following comparisons of equations:

(1) *Comparison of Eqs. 34 and 35 in Ref. (Zerimeche et al., 2023) with Eqs. 30 and 31 in Ref. (Choi, 2022a)*: Let us compare the angular frequencies Ω_i of the transformed system in Ref. (Zerimeche et al., 2023) with those in Ref. (Choi, 2022a), which are $\varpi_{0,i}$. Note that, when comparing them, $\eta(t)$ in their manuscript is the same as $2\delta(t)$ in Ref. (Choi, 2022a), which is a coefficient of the invariant operator (you can see this fact by comparing Eq. (38) with Eq. (2)). That is, both of them constitute the coefficients of the coupling term $\hat{x}_1\hat{x}_2$ in the invariant operator.

That comparison can be done from:

$$\tilde{\Omega}_1^2 = \delta_1\cos^2\frac{\theta}{2} + \delta_2\sin^2\frac{\theta}{2} + \frac{\eta}{2}\sqrt{\alpha_1\alpha_2}\sin\theta, \quad \text{(Eq. 34 in Ref. (Zerimeche et al., 2023))}$$

$$\tilde{\Omega}_2^2 = \delta_1\sin^2\frac{\theta}{2} + \delta_2\cos^2\frac{\theta}{2} - \frac{\eta}{2}\sqrt{\alpha_1\alpha_2}\sin\theta, \quad \text{(Eq. 35 in Ref. (Zerimeche et al., 2023))}$$

$$\varpi_{0,1}^2 = \omega_{0,1}^2\cos^2\varphi + \omega_{0,2}^2\sin^2\varphi + \delta\sqrt{\alpha_1\alpha_2}\sin(2\varphi), \quad \text{(Eq. 30 in Ref. (Choi, 2022a))}$$

$$\varpi_{0,2}^2 = \omega_{0,1}^2\sin^2\varphi + \omega_{0,2}^2\cos^2\varphi - \delta\sqrt{\alpha_1\alpha_2}\sin(2\varphi), \quad \text{(Eq. 31 in Ref. (Choi, 2022a))}$$

where δ_1 and δ_2 correspond to $\omega_{0,1}^2$ and $\omega_{0,2}^2$ in Choi's paper respectively, whereas θ corresponds to 2φ as mentioned earlier.

(2) *Comparison of the coefficient of $\hat{x}_1\hat{x}_2$ in the transformed invariant operator, which is given in the last line of Eq. 31 in Ref. (Zerimeche et al., 2023), with Eq. 32 in Ref. (Choi, 2022a)*: We can carry out this comparison such that

$$\text{Coefficient} = \frac{1}{2}\left(\eta\sqrt{\alpha_1\alpha_2}\cos\theta - (\delta_1-\delta_2)\sin\theta\right), \quad \text{(The last line in Eq. 31 in Ref. (Zerimeche et al., 2023))}$$

$$\bar{\delta} = M\left(\delta\sqrt{\alpha_1\alpha_2}\cos(2\varphi) - \frac{1}{2}(\omega_{0,1}^2-\omega_{0,2}^2)\sin(2\varphi)\right). \quad \text{(Eq. 32 in Ref. (Choi, 2022a))}$$

If we consider that Zerimeche *et al*. have chosen the arbitrary constant M as unity as mentioned earlier, the above two equations coincide with each other.

(3) *Comparison of Eq. 36 in Ref. (Zerimeche et al., 2023) and Eq. 33 in Ref. (Choi, 2022a)*: This comparison is as follows.

$$\tan\theta = \frac{\eta\sqrt{\alpha_1\alpha_2}}{\delta_1-\delta_2}, \quad \text{(Eq. 36 in Ref. (Zerimeche et al., 2023))}$$

$$\varphi = \frac{1}{2}\mathrm{atan}\left((\omega_{0,1}^2-\omega_{0,2}^2)/2, \delta\sqrt{\alpha_1\alpha_2}\right). \quad \text{(Eq. 33 in Ref. (Choi, 2022a))}$$

If we write Eq. 33 in Ref. (Choi, 2022a) in another form without loss of generality as

$$\tan(2\varphi) = \frac{2\delta\sqrt{\alpha_1\alpha_2}}{\omega_{0,1}^2-\omega_{0,2}^2}, \tag{42}$$

this exactly coincides with Eq. 36 in Ref. (Zerimeche et al., 2023).

From these comparisons, you can see no differences between theirs and Choi's development of the theory except for some notation differences that do not affect results. Their methods and results in Ref. (Zerimeche et al., 2023) are nothing but the ones given in Ref. (Choi, 2022a).

3.1.3. Additional remarks

(1) *Further scrutinizing their theory*: In addition to the above consequences, let us see another part of the work of Zerimeche *et al*. They argue that the coefficients of the invariant operator $\hat{I}_Z(t)$ should satisfy the following equation:

$$\alpha_i(t)\gamma_i(t) - \beta_i^2(t) = \delta_i, \tag{43}$$

where δ_i are real constants that we have seen above. However, this consequence is already appeared in Choi's previous several papers with complete verifications (see Eq. 15 of Ref. (Choi, 2003b), Eq. 17 of Ref. (Choi, 2003c), and Eq. 26 of Ref. (Choi, 2022a)). This fact together with our former two explanations means that there is no theory of them, which was developed throughout Ref. (Zerimeche et al., 2023).

(2) *Inconsistencies of their remark*: Zerimeche *et al*. mentioned in the introduction part of Ref. (Zerimeche et al., 2023) that they used the method of Ref. 36. Reference 36 in Ref. (Zerimeche et al., 2023) is the work of Macedo and Guedes, where it is given in Ref. (Macedo and

Guedes, 2012) in this paper. The authors of Ref. 36 performed two canonical transformations at first, in order to decouple the Hamiltonian. And then they introduced an invariant operator for the decoupled Hamiltonian. Finally, the time dependence of the decoupled Hamiltonian was removed by a unitary transformation.

However, the method of Zerimeche *et al*. is quite different from the one described above unlike their mention in Ref. (Zerimeche et al., 2023). They introduced an invariant operator at first (that is, without preliminary canonical transformations for decouplement of the Hamiltonian). And then, they performed two-step unitary transformation for the invariant operator using the unitary operators of which formulae were shown previously in comparison with Choi's. We confirm that this method is far from the method of Ref. 36, but very the same as the method adopted by Choi in Ref. (Choi, 2022a). However, they never say that they used Choi's method of Ref. (Choi, 2022a). Their such standpoint combined with the fact that the submission date of Ref. (Zerimeche et al., 2023) is before that of Ref. (Choi, 2022a) may mislead readers in a way that Zerimeche *et al*. have discovered the method of solving the problem prior to Choi's report.

It is now well-known that geometrical phase was discovered by Berry from his seminal paper, which is Ref. (Berry, 1984). Although Simon's paper (Simon, 1983) that analyzed the geometric phase was published prior to Berry's one at that time, Simon made clear the fact that the geometrical phase had been discovered by Berry (not by him) when he wrote that paper. This is in contrast to the fact that the authors of Ref. (Zerimeche et al., 2023) did not mention through that paper that Choi is the first researcher that solved the problem that they managed in it, even if it causes a misunderstanding.

At any rate, the theory in the whole part of section 2.2 in Ref. (Zerimeche et al., 2023) is nothing but the same as Choi's one given in Ref. (Choi, 2022a). Because there is no theory of them developed throughout Ref. (Zerimeche et al., 2023), the whole theory in it is actually Choi's one given in Ref. (Choi, 2022a).

3.2. Problems occurred in adopting Choi's theory

Choi's original theory in this context is complete as a matter of fact. However, many problems have occurred when Zerimeche *et al*. adopted it in their paper as we show in this section from now on.

3.2.1. Check for their understanding of taken system(s)

Prior to seeing their main mistakes in Ref. (Zerimeche et al., 2023), let us first check how they exactly understand the system(s) that they treated. In the title of Ref. (Zerimeche et al., 2023), Zerimeche *et al*. represented their system(s) as "a general time-dependent coupled oscillator". This may mean that they treated a general oscillator that is coupled itself. However, coupling between oscillators can be done between at least two oscillators. On one hand, they represented their system(s) as "a two-dimensional (2D) time-dependent coupled oscillator" in both abstract and introduction parts.

From the above fact, what the system(s) that Zerimeche *et al*. managed through Ref. (Zerimeche et al., 2023) is very obscure. That is, it is unclear whether they treated "general time-dependent coupled oscillators" given in Refs. (Choi, 2022a; Macedo and Guedes, 2012; Tobalina et al., 2020; Zhang et al., 2002) or "a 2D general time-dependent oscillator" given in Refs. (Fiore and Gouba, 2011; Mahdifar et al., 2012; Patra, 2023; Choi, 2003a). While it is questionable whether they fully understood the system(s) that they considered or not, they are perhaps unable to distinguish between "coupled oscillators" and "2D description of an oscillator".

3.2.2. Various non-negligible mistakes and errors

Zerimeche *et al*. have made many mistakes and errors when they adopted the theory of Choi. They are as follows.

(1) *Invalidity of the invariant operator*: Zerimeche *et al*. argue that $\eta(t)$ should satisfy the following two equations (see Eqs. 8 and 15 in Ref. (Zerimeche et al., 2023)):

$$\eta(t) = c_3(t)\alpha_1(t)m_1(t) \;\left(= c_3(t)\alpha_2(t)m_2(t)\right)\; [\equiv \eta_1(t)], \tag{44}$$

$$\eta(t) = -\int^t c_3(t')\, m_1 \rho_1\left(\dot{\rho}_1 + \rho_1\frac{\dot{\rho}_2}{\rho_2}\right)dt' \;[\equiv \eta_2(t)]. \tag{45}$$

The first equation among the above two has been rearranged without loss of generality for convenience of understanding. Because these two equations are independent of each other, there exist no $\eta(t)$ that satisfies them simultaneously. Hence, the construction of the invariant operator in their study is invalid.

To see in more detail for this, let us compare the time evolution of $\eta_1(t)$ and $\eta_2(t)$ with each other for a particular choice of parameters, where $\eta_1(t)$ and $\eta_2(t)$ are the first and second formulae of $\eta(t)$ as defined in Eqs. (44) and (45). We choose a simple case of time evolutions for parameters for this purpose, such that

$$m_i(t) = m_{0,i}\exp(\kappa_i t), \tag{46}$$

$$c_i(t) = c_{0,i}\exp(\kappa_i t), \tag{47}$$

$$c_3(t) = c_{0,3}(1+\kappa t), \tag{48}$$

$$\rho_i(t) = \rho_{0,i}\exp(-\kappa_i t/2), \tag{49}$$

$$\alpha_i(t) = \rho_{0,i}^2\exp(-\kappa_i t), \tag{50}$$

$$\delta_i(t) = m_{0,i}\rho_{0,i}^4(c_{0,i} - m_{0,i}\kappa_i^2/4), \tag{51}$$

where $m_{0,i}$, $c_{0,i}$, $c_{0,3}$, $\rho_{0,i}$, κ_i, and κ are real constants, while $i = 1, 2$. This choice satisfies the auxiliary equation for $\rho_i(t)$ given in Eq. 13 of Ref. (Zerimeche et al., 2023). Then we can write Eq. (45) after integrating in the form

$$\eta_2(t) = \frac{1}{2}c_{0,3}m_{0,1}\rho_{0,1}^2(\kappa_1+\kappa_2)\left(t + \frac{\kappa}{2}t^2\right) + \eta_{0,2}, \tag{52}$$

where $\eta_{0,2}$ is an integral constant.

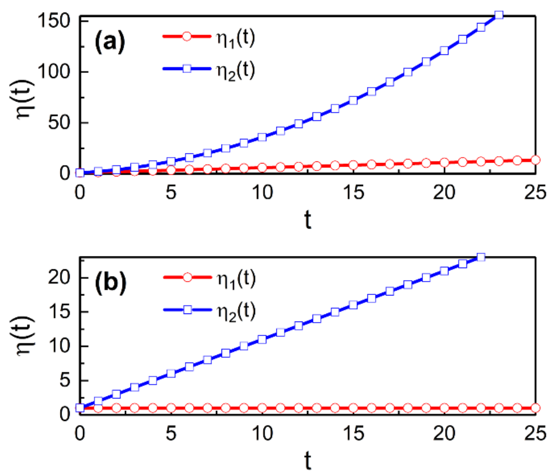

Figure 1: Graphical demonstration of the fact $\eta_1(t) \neq \eta_2(t)$. (a) is the time evolutions of $\eta_1(t)$ and $\eta_2(t)$ for the case where $m_{0,1} = 0.5$, $m_{0,2} = 1$, $c_{0,1} = c_{0,2} = 1$, $c_{0,3} = 1$, $\rho_{0,1} = \sqrt{2}$, $\rho_{0,2} = 1$, $\kappa_1 = \kappa_2 = 1$, and $\kappa = 0.5$, while $\eta_{0,2} = 1$ is imposed for keeping the initial condition as $\eta_1(0) = \eta_2(0)$. All values of parameters are taken to be dimensionless for convenience. (b) is the same as (a), but for $\kappa = 0$.

The time evolutions of $\eta_1(t)$ and $\eta_2(t)$ in this case are compared in Fig. 1 based on Eqs. (44) and (52) respectively. We see from this figure that $\eta_1(t)$ and $\eta_2(t)$ are quite different from each other. They are different even when $\kappa = 0$ (see Fig. 1(b)). It is not difficult to check that other choices for the types of parameters also give different manner between the evolutions of $\eta_1(t)$ and $\eta_2(t)$. Hence Zerimeche *et al*. did not provide a correct invariant operator. To solve this problem, they must impose a certain condition after Eq. 15 in Ref. (Zerimeche et al., 2023), like the condition appeared in Eq. 19 in Ref. (Choi, 2022a). Since they did not do in that way, $\eta_1(t)$ and $\eta_2(t)$ are different from each other and, hence, the invariant operator that they have suggested is in fact not an invariant operator. For this reason, their subsequent development of logic starting from their invariant operator (that is, from Eq. 16 in Ref. (Zerimeche et al., 2023)) is meaningless.

By the way, we suspect that the relevance of $\eta(t)$ on $c_3(t)\alpha_i(t)m_i(t)$, which appeared in the first equation among the two equations, Eqs. (44) and (45), was hinted by Eqs. 11 and 12 in Ref. (Choi, 2022a), which are Eqs. (6) and (7) in the present paper: notice that, in Eq. (6), $\delta(t)$ is identical to $\eta(t)/2$ in their notation as mentioned earlier, and $d(t)$ is identical to $c_3(t)/2$.

(2) *Misled unitary transformation*: The position of $\Sigma_{i=1}^2$ in $U_{Z,1}^B$ in the paper of Zerimeche *et al*. (Eq. 22 in Ref. (Zerimeche et al., 2023)) is different from that in Choi's paper. Hence, if we evaluate the whole transformation of the invariant operator in their work, it does not end up the form of the SHO invariant which is the same as the SHO Hamiltonian, but a very different one. This may be their mistake committed when they mimicked Choi's theory in theirs, since its wrong position naturally leads to an unsuitable result.

We have shown, in Appendix A, the detailed evaluation of the overall unitary transformation based on their formulae of the unitary operators. From this Appendix, we can confirm how much their actual outcome of the unitary transformation deviates from the decoupled SHO invariant that they intended to take. However, they say that they decoupled the invariant operator of the system in a scientifically coherent manner for the whole subject.

(3) *Missetting of equation*: Equation 27 in Ref. (Zerimeche et al., 2023) is composed of two parts as

$$\hat{U}_{Z,1}\hat{p}_i\hat{U}^{+}_{Z,1} = \frac{1}{\sqrt{\alpha_i}}, \qquad \hat{p}_i - \frac{\beta_i}{\sqrt{\alpha_i}}\hat{x}_i, \tag{53}$$

where $\hat{U}_{Z,1} = \hat{U}^{A}_{Z,1}\hat{U}^{B}_{Z,1}$. Here, the first part is wrong and the second part is meaningless.

(4) *Incorrect formula of the solutions*: Let us see Eq. 17 in Ref. (Zerimeche et al., 2023), which is

$$\hat{I}_Z(t)\,|\,\varphi_{n_1,n_2}\rangle = \lambda_{n_1,n_2}\,|\,\varphi_{n_1,n_2}\rangle. \tag{54}$$

Here, φ_{n_1,n_2} are irrelevant to φ appeared in the representation of $\hat{U}_2$ (see Eq. 28 in Ref. (Choi, 2022a)). Zerimeche *et al*. defined the eigenstates of $\hat{I}_Z(t)$ as $|\,\varphi_{n_1,n_2}\rangle$ as can be seen from the above equation. They represented their evaluated formula of these eigenstates in Eq. 38 in Ref. (Zerimeche et al., 2023). It is given by

$$|\,\varphi_{n_1,n_2}\rangle = \prod_{i=1}^{2}\left(\frac{\sqrt{\tilde{\Omega}_i}}{(\pi\hbar)^{1/2}n_i!2^{n_i}}\right)^{1/2} H_{n_i}\left(\sqrt{\frac{\tilde{\Omega}_i}{\hbar}}x_i\right)\exp\left(-\frac{i\tilde{\Omega}_i}{2\hbar}x_i^2\right). \tag{55}$$

The problem is that this is very different from Choi's formula of eigenstates, Eq. (17), which was proved to be correct from Sec. 2. Hence, we see that Zerimeche *et al*. did not provide accurate eigenstates of $\hat{I}_Z$.

In fact, Eq. 38 in Ref. (Zerimeche et al., 2023) is similar to the eigenstates of $\hat{I}''_Z$ instead of $\hat{I}_Z$, where $\hat{I}''_Z$ is the transformed invariant operator that they suggested and is of the form (see Eq. 33 in Ref. (Zerimeche et al., 2023))

$$\hat{I}''_Z(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\hat{p}_i^2 + \tilde{\Omega}_i^2(t)\hat{x}_i^2\right]. \tag{56}$$

However, they are not exactly the eigenstates of $\hat{I}''_Z$ if we consider that $\hat{I}''_Z$ is not equivalent to that of the time-independent SHO. That is, $\hat{I}''_Z$ is expressed in terms of $\tilde{\Omega}_i$ that are actually dependent on time. We see from Eqs. 34 and 35 in Ref. [Zerimeche et al., 2023] that $\tilde{\Omega}_i$ are represented in terms of θ which is dependent on time (see the subsequent subsection, Subsec. 3.2.3, in this paper). This is the reason why we represented the angular frequencies $\tilde{\Omega}_i(t)$ in Eq. (56) in time-variable forms, although Zerimeche *et al*. omitted the time dependencies when they represented them in Eq. 33 of Ref. (Zerimeche et al., 2023). It is well known that the eigenstates in this case are different from those of the textbook SHO (Khandekar and Lawande, 1975; Malkin et al., 1970).

In what follows, it seems that they did not know how to distinguish the eigenstates of $\hat{I}_Z$ and $\hat{I}''_Z$: this conclusion can be deduced from the fact that the notations $|\,\varphi_{n_1,n_2}\rangle$ in Eq. (55) are the same as $\hat{I}_Z$'s eigenstates given in Eq. (54), while they explained them as the eiegnstates of $\hat{I}''_Z$ in Ref. (Zerimeche et al., 2023) (see lines 14-16 on page 6 in Ref. (Zerimeche et al., 2023)). The eigenstates of $\hat{I}_Z$ are apparently different from those of $\hat{I}''_Z$ and they can be obtained from the inverse transformation of the eigenstates of $\hat{I}''_Z$, while Zerimeche *et al*. did not carry out such an inverse-way transformation.

(5) *Inadequacy of notation*: As we have seen above, Zerimeche *et al*. wrote the eigenstates as $|\,\varphi_{n_1,n_2}\rangle$ in Eq. 38 of Ref. (Zerimeche et al., 2023). However, since the eigenstates given in that equation belong to those represented in x_i spaces, they should write them as $\langle x_1, x_2\,|\,\varphi_{n_1,n_2}\rangle$ or as $\varphi_{n_1,n_2}(x_1, x_2, t)$. This is not just a problem of choosing a notation and hence their expression is actually wrong.

If we write $\langle x_1, x_2\,|$ as $\langle x_2\,|\langle x_1\,|$, $\langle x_i\,|$ have the dimension of $L^{-1/2}$ where L means length: this fact can be confirmed through the dimensional analysis from the well-known identity $\int_{-\infty}^{\infty} dx_i\,|\,x_i\rangle\langle x_i\,| = \mathrm{I}$ where I is unity. Since $\langle x_1, x_2\,|$ is not a dimensionless quantity, the dimensions of the left and right hand sides of Eq. 38 in Ref. (Zerimeche et al., 2023) are different from each other.

(6) *Unphysicality of the eigenstates*: Let us suppose the followings:

i). Ω_i in Ref. (Zerimeche et al., 2023) are independent of time.

ii). $|\varphi_{n_1,n_2}\rangle$ in Eq. 38 of Ref. (Zerimeche et al., 2023) are the eigenstates of $\hat{I}_Z''$ (instead of $\hat{I}_Z$) although these notations are the same as the ones in Eq. 17 of Ref. (Zerimeche et al., 2023), which is the eigenvalue equation of $\hat{I}_Z$ (not $\hat{I}_Z''$).

iii). The expressions $|\varphi_{n_1,n_2}\rangle$ in Eq. 38 of Ref. (Zerimeche et al., 2023) were replaced by the correct ones which are $\langle x_1, x_2 | \varphi_{n_1,n_2}\rangle$.

Even under these suppositions, Eq. 38 in Ref. (Zerimeche et al., 2023) is not free from faults. Zerimeche *et al*. represented the last part of that equation as an exponential function of an imaginary factor. However, this should be replaced with an exponentially decreasing function with x_i^2 as

$$\sim \exp\left(-\frac{i\tilde{\Omega}_i}{2\hbar}x_i^2\right) \quad \rightarrow \quad \sim \exp\left(-\frac{\tilde{\Omega}_i}{2\hbar}x_i^2\right). \tag{57}$$

We know that the absolute square of the wave functions (or eigenstates) is physically meaningful rather than wave functions themselves. Let us see absolute square of the above two factors with a limit $x_i \rightarrow \infty$:

$$\lim_{x_i\rightarrow\infty}\left|\exp\left(-\frac{i\tilde{\Omega}_i}{2\hbar}x_i^2\right)\right|^2 = 1 \qquad \text{(result of Ref. (Zerimeche et al., 2023))}, \tag{58}$$

$$\lim_{x_i\rightarrow\infty}\left|\exp\left(-\frac{\tilde{\Omega}_i}{2\hbar}x_i^2\right)\right|^2 = 0 \qquad \text{(result of the corrected case)}. \tag{59}$$

We see that, while Eq. (59) is being zero, Eq. (58) does not vanish. Hence, the oscillator can go an infinite region in the case of the solutions of Zerimeche *et al*., while this behavior is physically not allowed.

(7) *Overlooking of eigenvalues in the solutions*: The complete solutions of Eq. (54) also require λ_{n_1,n_2} as well as $\langle x_1, x_2 | \varphi_{n_1,n_2}\rangle$. However, we cannot see them throughout Ref. (Zerimeche et al., 2023), while they are clearly provided in the case of Ref. (Choi, 2022a) (see Eq. 41 in Ref. (Choi, 2022a)), where their formulae are also shown in Eq. (20) in the present paper.

In summary, Zerimeche *et al*. tried to solve the eigenvalue equations, Eq. (54), in the configuration spaces in Ref. (Zerimeche et al., 2023). They used even the unitary transformation technique for that purpose in a manner known by Choi. That is, they clearly said in section 2.2 in their paper that they introduce the unitary transformation in order to solve Eq. (54). However, they neither show correct eigenstates $\langle x_1, x_2 | \varphi_{n_1,n_2}\rangle$ nor provide eigenvalues λ_{n_1,n_2} which are two factors as the required solutions of that equation. This is the bare status of solutions that they derived.

(8) *Omitting of important condition*: Besides lots of non-negligible mistakes in the paper of Zerimeche *et al*. listed above, they made a very important mistake: the invariant operator proposed by Choi in Ref. (Choi, 2022a) is constant under the condition that $d(t)$ (or $c_3(t)/2$ in terms of the notation of Zerimeche *et al*.) yields Eq. (15). Zerimeche *et al*. not only failed to mention this condition, but nor cited it from the Choi's paper as well: due to this, the invariant operator in Ref. (Zerimeche et al., 2023) is not a time-constant, leading to invalidity of its content. This mistake of them may have arisen by their abrupt use of Choi's theory during the review process for the publication of Ref. (Zerimeche et al., 2023) in Modern Physics Letters B (MPLB).

3.2.3. Further important problem: another inadequate citation

We have pointed out an inadequate citation of Zerimeche *et al*. previously: that is, they mentioned that they adopted the method of Macedo and Guedes (Macedo and Guedes, 2012) in the development of the relevant theory in Ref. (Zerimeche et al., 2023), while actually they followed the method of Choi (Choi, 2022a), especially in section 2.2 in their paper. We would like to say their another inadequate citation here. After Eq. 36 of Ref. (Zerimeche et al., 2023), they wrote: "where

$$\theta = \arctan\left(\eta\sqrt{\alpha_1\alpha_2}\,[\delta_1-\delta_2]^{-1}\right), \tag{60}$$

is time-independent, to be convinced it is enough to follow the calculation method of $\partial\theta/\partial t$ developed in Ref. 38." Here, Ref. 38 in their paper is Choi's paper which is Ref. (Choi, 2022a) in this paper. The evaluation of $\partial\theta/\partial t$ is not difficult and hence anyone who knows basic mathematics can do it. That is, any mathematical technique in Ref. 38 is not required in that evaluation, which means that they cited it inadequately. To see in more detail for this, let us evaluate it directly now. Using the relation

$$\frac{d\arctan(y)}{dy} = \frac{1}{1+y^2}, \tag{61}$$

we can easily obtain the time derivative of θ in the form

$$\frac{\partial\theta}{\partial t} = \frac{(\delta_1-\delta_2)\{\eta(t)[\alpha_1(t)\dot{\alpha}_2(t)+\dot{\alpha}_1(t)\alpha_2(t)]+2\alpha_1(t)\alpha_2(t)\dot{\eta}(t)\}}{2\sqrt{\alpha_1(t)\alpha_2(t)}\,[(\delta_1-\delta_2)^2+\eta^2(t)\alpha_1(t)\alpha_2(t)]} \neq 0. \tag{62}$$

As you can see, any special technique, except for the relation in Eq. (61), is not used/required in this evaluation, contrary to their assertion. Meanwhile, another notable problem of Ref. (Zerimeche et al., 2023) is that this result is not zero, implying that θ varies over time depending on $\alpha_j(t)$ and $\eta(t)$. Equation (62) is also not zero even when $\dot{\eta}(t) = 0$, clearly showing the wrongness of their claim. For a more detailed proof that θ is not a time constant, we depicted the time evolution of θ directly in Fig. 2. We see that θ varies either we choose $\eta_1(t)$ or $\eta_2(t)$ as the formula of $\eta(t)$, where $\eta_1(t)$ and $\eta_2(t)$ were suggested by them as we have seen previously.

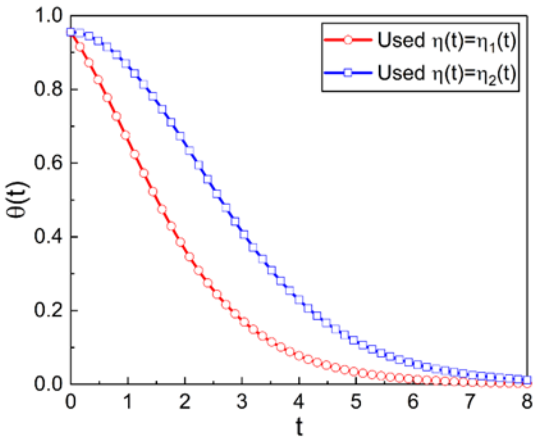

Figure 2: Time evolutions of θ for the case where Eqs. (46)-(51) are used for the time variation of parameters. $\eta(t) = \eta_1(t)$ is applied for the red curve, whereas $\eta(t) = \eta_2(t)$ for the blue curve. This is the graphical verification that θ in Ref. (Zerimeche et al., 2023) is not a time constant. All taken values of parameters are the same as those in Fig. 1(a). The whole parameters are chosen dimensionlessly.

The citation described in the above paragraph is the only citation of Ref. 38 throughout Ref. (Zerimeche et al., 2023) while we have shown that how such a citation is inadequate. By the way, Zerimeche *et al*. should cite in such a way that "the present problem was completely solved in Ref. 38" instead of that citation. At any rate, we can see no such a citation throughout Ref. (Zerimeche et al., 2023). They treated section 2.2. in Ref. (Zerimeche et al., 2023) as if it is the result of their theory. Their omission of the mentioning that "the problem had been solved previously by Choi" may mislead readers into thinking the problem was resolved by them. Despite the fact that Choi's paper was originally complete, that is, no further explanation was needed, they rehashed the content of Choi's paper and published it with lots of wrong statements and incorrect mathematical outcomes due to their numerous mistakes in the process. Hence, the unnecessary publication of their paper did not contribute to broadening the academic horizon, but rather caused many confusion in academia.

3.3. Examining possibilities in replacing their theory with Choi's one

The submission date of Ref. (Zerimeche et al., 2023) in MPLB is 12 Jul. 2022 and Choi's work, Ref. (Choi, 2022a), was uploaded in arXiv on 14 Oct. 2022. Regarding the fact that Ref. (Zerimeche et al., 2023) had been submitted in MPLB before Choi threw Ref. (Choi, 2022a) open to the public, we can check the validity of the following three cases of our supposition.

i). **Case 1**: Zerimeche *et al*. knew a correct theory of solving quantum problem of the system described by the Hamiltonian given in Eq. (36) based on the invariant operator theory at the initial time that they submitted Ref. (Zerimeche et al., 2023) in MPLB and their theory was the same as Choi's one.

<Check>: *In this case, they might have insisted in such a way that they had found the correct theory before Choi discovered it from Ref. (Choi, 2022a). However, because they did not insist in such a way from Ref. (Zerimeche et al., 2023), this case is not the case*.

ii). **Case 2**: Zerimeche *et al*. knew a correct theory of solving the quantum problem based on the invariant operator theory at the initial time that they submitted their paper in MPLB and their theory, except for the consequence, was different from Choi's one.

<Check>: *In this case, they might have insisted that their theory and Choi's theory are different from each other, but both are correct. However, they did not insist in such a way from Ref. (Zerimeche et al., 2023), this case is also not the case*.

iii). **Case 3**: Zerimeche *et al*. did not know a correct theory of solving the quantum problem based on the invariant operator theory at the initial time that they submitted their paper in MPLB, but they abruptly abandon their wrong theory and adopted Choi's theory soon after it had been opened by him during the review process of their manuscript.
<Check>: *This case is possible if we consider the circumstances, together with scrutinizing the arguments in the content of Ref. (Zerimeche et al., 2023).*

When we examining the above three cases, we have also considered the resubmission date of the revised version of Ref. (Zerimeche et al., 2023) as well as the initial submission dates of the two papers, Refs. (Choi, 2022a) and (Zerimeche et al., 2023). Among the above three supposed cases, **case 3** is only possible. From this, we conclude that Zerimeche *et al*. might not know how to solve the problem in their paper and hence the content of their manuscript might be wrong at the time that they initially submitted it in MPLB. But they may abruptly abandon their wrong theory when Choi opened Ref. (Choi, 2022a) to public during the review process of Ref. (Zerimeche et al., 2023) and, at the same time, they may have replaced their theory with Choi's one. The resubmission date of the revised version of Ref. (Zerimeche et al., 2023) is 2 Dec. 2022 and this date is about two months later than the submission date of Ref. (Choi, 2022a) which is 14 Oct. 2022: this fact supports our explanation, whereas we can find no other possible explanation for the appearance of the broad overlapping part in the two papers. Notably, in spite of such a replacement via the correct theory of Choi, their development of the quantum theory is still not right but seriously problematic on account of their numerous mistakes and blunders in adopting Choi's theory as can be seen from Sec. 3.2.

Even when they received positive response from some reviewers in the review process of the paper, the authors are not free from their responsibility regarding the content of their paper because the reviewers cannot consider all things when they write their reports. This is also the reason why it is hard to ask reviewers' mistakes to reviewers.

4. RE-ESTIMATING ZERIMECHE ET *AL*.'S WORK BASED ON THE COMPLETE SOLUTIONS OF CHOI'S WORK

4.1. Re-estimation based on Choi's works Refs. (Choi, 2022a) and (Choi, 2022b)

The two verifications in Sec. 2, which are the accuracy of the invariant operator and the exactness of quantum solutions, cannot be done if we do not use Eq. (15) that was failed to mention in Ref. (Zerimeche et al., 2023). On the other hand, Ref. (Zerimeche et al., 2023) provides neither correct wave functions (see items (4), (5), and (6) in Subsec. 3.2.2) nor exact eigenvalues (see item (7) in Subsec. 3.2.2), setting aside the incorrectness of the invariant operator (see item (1) in Subsec. 3.2.2), though they tried to copycat the correct ones in Ref. (Choi, 2022a).

In particular, they made three mistakes only in the eigenstates given in Eq. (55). They are as follows:

(1) The notations $|\varphi_{n_1,n_2}\rangle$ in Eq. (55) are the same as the eigenstates given in Eq. (54), which need inverse transformation in their derivation. But they did not perform the inverse transformation when they obtained them.

(2) The correct notations of the eigenstates of $\hat{I}_Z$ in the configuration space are $\langle x_1, x_2 | \varphi_{n_1,n_2}\rangle$. However, they used $|\varphi_{n_1,n_2}\rangle$ in Eq. (55) instead of them as those notations. Because the dimensions of $\langle x_1, x_2 | \varphi_{n_1,n_2}\rangle$ and $|\varphi_{n_1,n_2}\rangle$ are different from each other, $\langle x_1, x_2 | \varphi_{n_1,n_2}\rangle$ cannot be replaced by $|\varphi_{n_1,n_2}\rangle$.

(3) The factors $\exp[-i\tilde{\Omega}_i x_i^2/(2\hbar)]$ in Eq. (55) do not vanish at $x_i \to \infty$, which means that they are physically unacceptable. They should be replaced by correct ones that exponentially decrease as $|x_i|$ increase.

Even when the above three mistakes are fixed, the eigenstates in Eq. (55) still not right. Because such eigenstates are obtained from inverse transformation of the eigenstates of $\hat{I}_Z''$, it should be claimed that the formula of $\hat{I}_Z''$ is right in order to guarantee correctness of them. However, we know from previous analysis that Zerimeche *et al*.'s expression of $\hat{I}_Z''$ is not right: recall that $\hat{I}_Z''$ in Ref. (Zerimeche et al., 2023) is wrong either when they use $\eta_1(t)$ or when $\eta_2(t)$ as the formula of $\eta(t)$.

Besides, Zerimeche *et al*. also managed three coupled oscillators in the same paper. They mentioned in the Abstract of Ref. (Zerimeche et al., 2023) as "The generalization to three-dimensional (3D) time-dependent coupled oscillator is briefly discussed where a diagonalized invariant, which is exactly the sum of three simple harmonic oscillators, is obtained under specific conditions on the parameters." As seen from this statement, they insisted that they obtained an exact diagonalizable invariant operator for time-dependent three coupled oscillators. However, we cannot find such an invariant throughout Ref. (Zerimeche et al., 2023). Where is the mathematical formula of that invariant including its detailed evaluation? If there is, is it their own one? An original and seminal one? Even when we admit that they obtained it in that paper, it may be useless because we also cannot find exact necessary conditions for establishing the invariant of time-dependent three coupled oscillators in Ref. (Zerimeche et al., 2023). This is in contrast to Choi's another recent work, so-called Ref. (Choi, 2022b) which is in fact a sister paper of Ref. (Choi, 2022a) and deals full quantum description

for time-dependent three coupled oscillators. Choi not only obtained the exact invariant operator for the three coupled time-varying systems, he derived associated complete quantum solutions beyond it in that paper. Choi showed that the construction of the invariant operator for that case also needs conditions similar to Eq. (15) for the two coupled case. That is, Choi suggested three such conditions in Eqs. 23-25 with Eqs. 26-28 in Ref. (Choi, 2022b), which were not represented in Ref. (Zerimeche et al., 2023).

In short, Zerimeche *et al*. tried to extend their adopted theory to three coupled oscillators, but they failed even to figure out the invariant operator contrary to what they argued in the Abstract in their paper. It may be clear that, for the case of three coupled systems, they could not see Choi's manuscript in the same subject, Ref. (Choi, 2022b), until they finished their last revision of Ref. (Zerimeche et al., 2023) for its resubmission to MPLB because Ref. (Choi, 2022b) was opened slightly later than their last revision date: that is, Ref. (Choi, 2022b) was submitted to arXiv on 15 Dec. 2022, which is about two weeks later than their last resubmission date of Ref. (Zerimeche et al., 2023), 2 Dec. 2022.

Through the analysis of Ref. (Zerimeche et al., 2023) in comparison with Ref. (Choi, 2022a) (and Choi's another work Ref. (Choi, 2022b)) up until now, we have shown that its content is very problematic as a scientific work. As a summary, the main points that we have concluded in that way are as follows:

i). *The dynamics for obtaining quantum solutions in Fock states for a more general system described by Eq. (37) was completely known by Choi through Ref. (Choi, 2022a). Choi's theory in that paper is based on the two-step unitary transformation method and exact. Zerimeche et al. also treated and solved the same problem for the system covered by Choi's development (Ref. (Choi, 2022a)) using two-step unitary transformation method which is the same as Choi's treatment. They provided no new ideas/theory related to solving the problem, while they knew the same problem had already been solved by Choi before they adopted Choi's theory appeared in Ref. (Choi, 2022a). Nevertheless, they wrote in Ref. (Zerimeche et al., 2023) as if the problem was solved by them by representing such that "In this paper, we investigate …" (line 24 on page 2 in Ref. (Zerimeche et al., 2023)), "In section 2, we evaluate the study of the time-dependent 2D coupled system and approach the whole subject in a scientifically coherent manner to show that the invariant operator of the system can be uncoupled …" (line 28 on page 2 in Ref. (Zerimeche et al., 2023)), "In order to solve the problem in a clearer way we adopt, …" (line 37 on page 2 in Ref. (Zerimeche et al., 2023)), "In order to solve the eigenvalues equation (17), we introduce …" (line 19 on page 4 in Ref. (Zerimeche et al., 2023)), and so on.*

ii). *Moreover, Zerimeche et al. represented the same outcome as Choi's one except for their mistakes and blunders in adopting Choi's theory in solving the problem*.

iii). *Zerimeche et al. mentioned that they modeled Ref. (Macedo and Guedes, 2012) of which method is not so much related to theirs, while they omitting the mentioning in the introduction part (or elsewhere) that the same problem had already been solved by Choi through the same unitary transformation method in spite of the fact that they used that method. Moreover, they wrote section 2.2. in Ref. (Zerimeche et al., 2023) as if it is the results of their theory*.

The research of the two papers, Refs. (Choi, 2022a) and (Zerimeche et al., 2023), were carried out in a similar period, although the quantum solutions which are the main subject of both papers were originally found by Choi. The submission date of Ref. (Zerimeche et al., 2023) is 12 Jul. 2022 and that of Ref. (Choi, 2022a) is 14 Oct. 2022 as mentioned earlier. If we merely think such submission dates, Ref. (Zerimeche et al., 2023) is three months earlier than Ref. (Choi, 2022a). For this reason, readers may misapprehend that the discovery of quantum solutions by Zerimeche et al. is before Choi's report in Ref. (Choi, 2022a). If we think of this fact, their taciturnity about which is former and which is later between Choi's discovery and theirs throughout Ref. (Zerimeche et al., 2023) is a nontrivial problem. Recall that Zerimeche et al. cited related papers inadequately.

iv). *The central part in Ref. (Zerimeche et al., 2023) of which content is largely shared with Ref. (Choi, 2022a) is section 2.2 as we have seen from Subsecs. 3.1.1 and 3.1.2 in the present paper. This problematic section, which is actually the same as the one in Ref. (Choi, 2022a), is too large beyond acceptable to the public*.

v). *The title of section 2.2 in Ref. (Zerimeche et al., 2023) is "Unitary transformations: Results and discussion". This title implies that they derived results from unitary transformation and discussed the results. We can confirm from that fact that section 2.2 which overlaps with Choi's one is the main consequence of Ref. (Zerimeche et al., 2023). Because the theory of section 2.2 is not theirs (but Choi's) as we have shown previously, the main consequence of Ref. (Zerimeche et al., 2023) is actually not theirs (but Choi's)*.

vi). *Another problem that cannot be overlooked is that Zerimeche et al.'s paper, Ref. (Zerimeche et al., 2023), is strictly a research paper (not a review paper) of which content must be their own work. This fact may additionally justify and augment the criticism mentioned in the above items.*

Zerimeche *et al*. may have tried to publish a wrong paper by submitting the original version of Ref. (Zerimeche et al., 2023) in MPLB, but, in the review process, they may have replaced their theory with Choi's one as soon as Choi's paper was opened to public by him. The fact that there is no theory in Ref. (Zerimeche et al., 2023), which Zerimeche *et al*. newly discovered through that paper, also supports this opinion. By the way, the finally arranged content of Ref. (Zerimeche et al., 2023) is still wrong, problematic, and inconsistent due to a large number of their blunders in the course of replacing their theory with Choi's one (see Subsecs. 3.2.2 and 3.2.3), letting alone the ambiguity of their representation for the system that they treated (see Subsec. 3.2.1).

Above all, their taciturnity throughout Ref. (Zerimeche et al., 2023) about the fact that Choi found the solutions of the system before they realized their solutions is quite troubling as has been pointed out earlier. We stress again that, because the submission date of Ref. (Zerimeche et al., 2023) is before that of Ref. (Choi, 2022a), such a taciturnity brings about misunderstanding of the originality of the

work, which acts as the source of misleading readers. Their inadequate citations may also give support to such a misleading. For this reason, we afraid that it is academically recognized that the complete quantum solutions of general time-dependent two coupled harmonic oscillators described by Eq. (36) were discovered by Zerimeche *et al*. for the first time, although Choi is the first discoverer of them as far as we know. Recall that Choi's solutions in Ref. (Choi, 2022a) cover corrected solutions of the ones in Ref. (Zerimeche et al., 2023).

Remember that, though Newton's gravitation theory is revolutionary in understanding physics and astronomy from a fundamental level, we know at the same time that we cannot exclude the possibility that that theory was not originated by him but by Hooke (Westfall, 1967; Guicciardini, 2020). We wish that our discussion about the problem of Ref. (Zerimeche et al., 2023) provides sure grounds for discriminating the fact that the quantum theory at hand was originally formulated by Choi (not by Zerimeche *et al*.). Additionally, we wish that this work serves as a momentum to rethink the responsibility of researchers when they carry out their own works.

Soon after the completion of Ref. (Choi, 2022a), Choi has also solved the same problem of general time-dependent three coupled oscillators in the draft of Ref. (Choi, 2022b). We think that both the Ref. (Choi, 2022a) and Ref. (Choi, 2022b) are important in unfolding quantum dynamics of general time-dependent coupled oscillators, because the complete quantum solutions for the system considered in each paper enable us to analyze their quantum characteristics precisely.

4.2. Other misleading arguments of Zerimeche *et al.*

We have seen the errors, mistakes, and blunders given in Ref. (Zerimeche et al., 2023) up until now. In this section we focus on analyzing misleadings in Zerimeche *et al*.'s other works together with examining how incorrect their outcomes are.

4.2.1. Suggestion of inadequate eigenvalues

Zerimeche *et al*. argued that the eigenvalues in Refs. (Choi, 2022b; Kronenburg, 2015; Denton et al., 2022; Hassoul et al., 2022) for a general 3×3 matrix, which have been known to be correct, are wrong and they suggested other eigenvalues as alternatives. To see about this in detail, let us regard the matrix that Choi treated in Ref. (Choi, 2022b) in diagonalizing a transformed invariant operator. It is of the form

$$\Gamma_M = \begin{pmatrix} \omega_{0,1}^2 & \Delta_{12} & \Delta_{13} \\ \Delta_{12} & \omega_{0,2}^2 & \Delta_{23} \\ \Delta_{13} & \Delta_{23} & \omega_{0,3}^2 \end{pmatrix}, \tag{63}$$

where $\omega_{0,j}$ $(j = 1, 2, 3)$ are angular frequencies and Δ_{kl} $(k = 1, 2;\ l = 2, 3)$ are coupling strengths for the transformed invariant operator. Though Choi adopted Eq. (63) as an *ad hoc* one for quantizing time-dependent three coupled oscillators, this matrix is a general type. Choi considered the following three eigenvalues of Γ_M to develop his quantum theory:

$$\varpi_{0,1}^2 = \frac{1}{3}[\omega_0^2 + \mathrm{J}\cos\Theta], \tag{64}$$

$$\varpi_{0,2}^2 = \frac{1}{3}[\omega_0^2 + \mathrm{J}\cos(\Theta - 2\pi/3)], \tag{65}$$

$$\varpi_{0,3}^2 = \frac{1}{3}[\omega_0^2 + \mathrm{J}\cos(\Theta + 2\pi/3)], \tag{66}$$

where $\omega_0 = (\omega_{0,1}^2 + \omega_{0,2}^2 + \omega_{0,3}^2)^{1/2}$, and

$$\mathrm{J} = \sqrt{2}\left[(\omega_{0,1}^2 - \omega_{0,2}^2)^2 + (\omega_{0,1}^2 - \omega_{0,3}^2)^2 + (\omega_{0,2}^2 - \omega_{0,3}^2)^2 + 6\Delta^2\right]^{1/2}, \tag{67}$$

$$\Theta = \frac{1}{3}\arccos\left(\frac{4A}{J^3}\right), \tag{68}$$

with

$$\Delta = (\Delta_{12}^2 + \Delta_{13}^2 + \Delta_{23}^2)^{1/2}, \tag{69}$$

$$\begin{aligned} A = {} & -3\,(\omega_{0,1}^2 + \omega_{0,2}^2)\,(\omega_{0,1}^2 + \omega_{0,3}^2)\,(\omega_{0,2}^2 + \omega_{0,3}^2) \\ & - 27\,(\omega_{0,1}^2\Delta_{23}^2 + \omega_{0,2}^2\Delta_{13}^2 + \omega_{0,3}^2\Delta_{12}^2) + 9\,\omega_0^2\,\Delta^2 \\ & + 2\,(\omega_{0,1}^6 + \omega_{0,2}^6 + \omega_{0,3}^6) + 18\,(\omega_{0,1}^2\,\omega_{0,2}^2\,\omega_{0,3}^2 + 3\Delta_{12}\Delta_{13}\Delta_{23}). \end{aligned} \tag{70}$$

The representation styles of the eigenvalues for the matrix in Eq. (63) are slightly different from paper to paper among Refs. (Choi, 2022b; Kronenburg, 2015; Hassoul et al., 2022), but they are mathematically the same as one another and identical to Eqs. (64)-(66).

Instead of the eigenvalues $\varpi_{0,j}^2$ above, Zerimeche *et al*. suggested their own eigenvalues in Ref. (Zerimeche et al., 2022a). They glossed over their eigenvalues, saying that "At the end of this section, let us note that the Hassoul *et al*'s [1] paper is largely inspired from the incoherent results of [3]" (see the last sentence on page 4 of Ref. (Zerimeche et al., 2022a)). Here, Refs. 1 and 3 that they have cited correspond to Refs. (Hassoul et al., 2022) and (Habarrih et al., 2021) in this work, respectively. They misled readers through this argument, to say nothing of embarrassing and annoying related researchers who used Eqs. (64)-(66) as the engenvalues, including the authors of Refs. (Hassoul et al., 2022) and (Habarrih et al., 2021). Both of Refs. (Hassoul et al., 2022) and (Habarrih et al., 2021) have used the eigenvalues of Eqs. (64)-(66).

Let us prove that Eqs. (64)-(66) are no problems at all. The exactness of them can be verified by examining the secular equation of the matrix Γ_M. We consider the following determinant for that purpose:

$$\mathcal{S} = \begin{vmatrix} \omega_{0,1}^2 - \lambda & \Delta_{12} & \Delta_{13} \\ \Delta_{12} & \omega_{0,2}^2 - \lambda & \Delta_{23} \\ \Delta_{13} & \Delta_{23} & \omega_{0,3}^2 - \lambda \end{vmatrix}, \tag{71}$$

where λ is an arbitrary eigenvalue of Γ_M given in Eq. (63). If the secular equation, $\mathcal{S} = 0$, is satisfied for a certain λ, the considered λ is a correct eigenvalue as is well known.

Let us take $\lambda = \varpi_{0,1}^2$ given in Eq. (64) for instance. Then, by expanding Eq. (71) after inserting ω_0 and Eqs. (67) and (68) in it, we have

$$\begin{aligned}\mathcal{S} = \frac{1}{27}\{&-A + 54\Delta_{12}\Delta_{13}\Delta_{23} + 9\Delta_{12}^2(\omega_{0,1}^2 + \omega_{0,2}^2 - 2\omega_{0,3}^2) \\ &+ 9\Delta_{13}^2(\omega_{0,1}^2 - 2\omega_{0,2}^2 + \omega_{0,3}^2) + 18(\Delta_{12}^2 + \Delta_{13}^2 + \Delta_{23}^2 - \Delta^2) \\ &\times[3\Delta^2 + \omega_{0,1}^4 + \omega_{0,2}^4 - \omega_{0,2}^2\omega_{0,3}^2 + \omega_{0,3}^4 - \omega_{0,1}^2(\omega_{0,2}^2 + \omega_{0,3}^2)]^{1/2} \\ &\times\cos\{3^{-1}\arccos\{A[2(3\Delta^2 + \omega_{0,1}^4 + \omega_{0,2}^4 - \omega_{0,2}^2\omega_{0,3}^2 + \omega_{0,3}^4 \\ &- \omega_{0,1}^2(\omega_{0,2}^2 + \omega_{0,3}^2))^{3/2}]^{-1}\}\} - (2\omega_{0,1}^2 - \omega_{0,2}^2 - \omega_{0,3}^2) \\ &\times[9\Delta_{23}^2 - (\omega_{0,1}^2 + \omega_{0,2}^2 - 2\omega_{0,3}^2)(\omega_{0,1}^2 - 2\omega_{0,2}^2 + \omega_{0,3}^2)]\}.\end{aligned} \tag{72}$$

From a minor evaluation after inserting Eqs. (69) and (70) into the above equation, we eventually arrive $\mathcal{S} = 0$. $\varpi_{0,1}^2$ is therefore an exact eigenvalue of Γ_M. It is possible to verify, in the same way, that other two eigenvalues, $\varpi_{0,2}^2$ and $\varpi_{0,3}^2$, are also exact.

Instead of Eqs. (64)-(66), Zerimeche *et al*. proposed the followings (see Eq. 21 in Ref. (Zerimeche et al., 2022a)) as the eigenvalues of the same Γ_M:

$$\lambda_1 = \frac{1}{3}[\omega_{0,1}^2 + \omega_{0,2}^2 + \omega_{0,3}^2 + 2(\Delta_{12} + \Delta_{13} + \Delta_{23})], \tag{73}$$

$$\lambda_2 = \frac{1}{3}[\omega_{0,1}^2 + \omega_{0,2}^2 + \omega_{0,3}^2 - (\Delta_{12} + \Delta_{13} + \Delta_{23})] + z_0, \tag{74}$$

$$\lambda_3 = \frac{1}{3}[\omega_{0,1}^2 + \omega_{0,2}^2 + \omega_{0,3}^2 - (\Delta_{12} + \Delta_{13} + \Delta_{23})] - z_0, \tag{75}$$

where

$$z_0 = [\Delta_{12}^2 + \Delta_{13}^2 + \Delta_{23}^2 - (\Delta_{12}\Delta_{13} + \Delta_{12}\Delta_{23} + \Delta_{13}\Delta_{23})]^{1/2}. \tag{76}$$

Let us also check whether these eigenvalues are accurate or not. If we take $\lambda = \lambda_1$ given in Eq. (73) as an example, Eq. (71) becomes

$$\begin{aligned}\mathcal{S}_Z = \frac{1}{27}\{&9\Delta_{12}^2[\omega_{0,1}^2 + \omega_{0,2}^2 - 2\omega_{0,3}^2 + 2(\Delta_{12} + \Delta_{13} + \Delta_{23})] \\ &+ 9\Delta_{13}^2[\omega_{0,1}^2 - 2\omega_{0,2}^2 + \omega_{0,3}^2 + 2(\Delta_{12} + \Delta_{13} + \Delta_{23})] \\ &+ 9\Delta_{23}^2[-2\omega_{0,1}^2 + \omega_{0,2}^2 + \omega_{0,3}^2 + 2(\Delta_{12} + \Delta_{13} + \Delta_{23})] \\ &-[\omega_{0,1}^2 + \omega_{0,2}^2 - 2\omega_{0,3}^2 + 2(\Delta_{12} + \Delta_{13} + \Delta_{23})] \\ &\times[\omega_{0,1}^2 - 2\omega_{0,2}^2 + \omega_{0,3}^2 + 2(\Delta_{12} + \Delta_{13} + \Delta_{23})] \\ &\times[-2\omega_{0,1}^2 + \omega_{0,2}^2 + \omega_{0,3}^2 + 2(\Delta_{12} + \Delta_{13} + \Delta_{23})] \\ &+ 54\Delta_{12}\Delta_{13}\Delta_{23}\} \neq 0.\end{aligned} \tag{77}$$

Though the above expression is somewhat complicated, it cannot be reduced further and is far from zero. Hence, it is apparent that λ_1 suggested by Zerimeche *et al*. is not an eigenvalue of Γ_M. It is not difficult to verify that their other two quantities, λ_2 and λ_3, are also not eigenvalues, while we have provided graphical demonstration of this fact in Fig. 3 for clarity. Based on this, we conclude that the related evaluations, such as diagonalization, eigenvectors, and transformations, which they have provided in Ref. (Zerimeche et al., 2022a) are not right and even meaningless.

It is hence clear that Zerimeche *et al*.'s assertion is wrong, despite their overwhelming criticism for other researchers' eigenvalues, $\varpi_{0,j}^2$, and related outcomes. They must keep in mind the fact that they are actually wrong even though they alleged the accuracy of their development in a manner of full assurance in Ref. (Zerimeche et al., 2022a). Their misleading of public in science does not cease here but continues. We show another example in the next subsection.

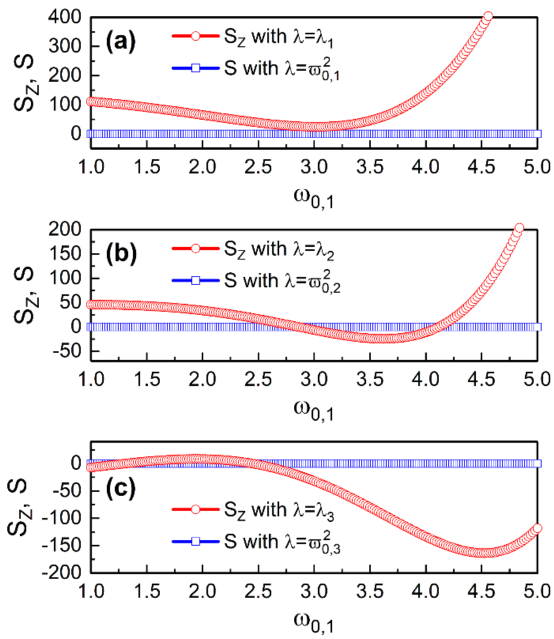

Figure 3: (a) depicts the variation of S_Z in Eq. (77) which is with the choice of $\lambda = \lambda_1$ and S in Eq. (72) which is with $\lambda = \varpi_{0,1}^2$ as a function of $\omega_{0,1}$. We also depicted the same graphics S_Z and S with the choice of $\lambda = \lambda_2$ and $\lambda = \varpi_{0,2}^2$ in (b), and with the choice of $\lambda = \lambda_3$ and $\lambda = \varpi_{0,3}^2$ in (c), respectively. We used $\omega_{0,2} = 2$, $\omega_{0,3} = 3$, $\Delta_{12} = 1$, $\Delta_{13} = 2$, and $\Delta_{23} = 3$. We see that S_Z is not zero, while S is always zero.

4.2.2. Another misleading transformation

Zerimeche *et al*. claimed something indescribably wrong from Ref. (Zerimeche et al., 2022b). Let us see it now. They proposed another frame of transformation (Zerimeche et al., 2022b) for the system of which Hamiltonian is the same as Eq. (36), but related to the following form of the invariant operator:

$$\hat{I}_Z(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\alpha_i(t)\hat{p}_i^2 + \beta_i(t)(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i) + \gamma_i(t)\hat{x}_i^2\right] + \frac{1}{2}\eta(t)\hat{x}_1\hat{x}_2, \tag{78}$$

where

$$\alpha_i(t) = \frac{1}{m_i(t)}, \tag{79}$$

$$\beta_i(t) = \frac{\dot{m}_i(t)}{2m_i(t)}, \tag{80}$$

$$\gamma_i(t) = \int_0^t \frac{c_i(t)\dot{m}_i(t)}{m_i(t)}dt, \tag{81}$$

$$\eta(t) = \int_0^t c_3(t)\left(\frac{\dot{m}_1(t)}{2m_1(t)} + \frac{\dot{m}_2(t)}{2m_2(t)}\right)dt. \tag{82}$$

The forms of $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$, and $\eta(t)$ represented above are different from those of $\hat{I}_Z(t)$ given in Eq. (38). Zerimeche *et al*. considered their own transformation for $\hat{\mathrm{I}}_Z(t)$ such that (see Eqs. 7 and 8 in Ref. (Zerimeche et al., 2022b)):

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{m_1(t)}}\cos\frac{\theta}{2} & -\frac{1}{\sqrt{m_1(t)}}\sin\frac{\theta}{2} \\ \frac{1}{\sqrt{m_2(t)}}\sin\frac{\theta}{2} & \frac{1}{\sqrt{m_2(t)}}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{pmatrix}, \tag{83}$$

$$\hat{p}_i = -m_i(t)\beta_i\hat{x}_i + \sqrt{m_i(t)}\hat{P}_i. \tag{84}$$

From this transformation, they obtained the transformed invariant operator as

$$\hat{\mathrm{I}}_Z(t) = \frac{1}{2}\sum_{i=1}^{2}\left(\hat{P}_i^2 + w_i^2\hat{Q}_i^2\right) + \Gamma(t)\hat{Q}_1\hat{Q}_2, \tag{85}$$

where

$$\begin{aligned} w_1^2(t) = {} & \left(\frac{\int_0^t [c_1\dot{m}_1/m_1]dt}{m_1} - \frac{\dot{m}_1^2}{4m_1^2}\right)\cos^2\frac{\theta}{2} + \left(\frac{\int_0^t [c_2\dot{m}_2/m_2]dt}{m_2} - \frac{\dot{m}_2^2}{4m_2^2}\right)\sin^2\frac{\theta}{2} \\ & + \left(\frac{\int_0^t c_3[\dot{m}_1/(2m_1) + \dot{m}_2/(2m_2)]dt}{2\sqrt{m_1 m_2}}\right)\sin\theta, \end{aligned} \tag{86}$$

$$\begin{aligned} w_2^2(t) = {} & \left(\frac{\int_0^t [c_1\dot{m}_1/m_1]dt}{m_1} - \frac{\dot{m}_1^2}{4m_1^2}\right)\sin^2\frac{\theta}{2} + \left(\frac{\int_0^t [c_2\dot{m}_2/m_2]dt}{m_2} - \frac{\dot{m}_2^2}{4m_2^2}\right)\cos^2\frac{\theta}{2} \\ & - \left(\frac{\int_0^t c_3[\dot{m}_1/(2m_1) + \dot{m}_2/(2m_2)]dt}{2\sqrt{m_1 m_2}}\right)\sin\theta, \end{aligned} \tag{87}$$

$$\begin{aligned} \Gamma(t) = {} & -\left[\left(\frac{\int_0^t [c_1\dot{m}_1/m_1]dt}{m_1} - \frac{\dot{m}_1^2}{4m_1^2}\right) - \left(\frac{\int_0^t [c_2\dot{m}_2/m_2]dt}{m_2} - \frac{\dot{m}_2^2}{4m_2^2}\right)\right]\sin\theta \\ & + \left(\frac{\int_0^t c_3[\dot{m}_1/(2m_1) + \dot{m}_2/(2m_2)]dt}{\sqrt{m_1 m_2}}\right)\cos\theta. \end{aligned} \tag{88}$$

Their result Eq. (85) is however wrong, whereas we have shown the exact result of the transformation with Eqs. (83) and (84) in Appendix B.

In addition, after choosing θ in the form

$$\begin{aligned} \tan\theta = {} & \frac{1}{\sqrt{m_1 m_2}}\int_0^t c_3[\dot{m}_1/(2m_1) + \dot{m}_2/(2m_2)]dt \\ & \times\left[\left(\frac{1}{m_1}\int_0^t [c_1\dot{m}_1/m_1]dt - \frac{\dot{m}_1^2}{4m_1^2}\right) - \left(\frac{1}{m_2}\int_0^t [c_2\dot{m}_2/m_2]dt - \frac{\dot{m}_2^2}{4m_2^2}\right)\right]^{-1}, \end{aligned} \tag{89}$$

Zerimeche *et al*. represented the invariant operator in terms of annihilation operators $\hat{a}_i$ (and creation operators $\hat{a}_i^{+}$ that are their Hermitian adjoints), which are defined as

$$\hat{a}_1 = \frac{1}{\sqrt{2\hbar w_1}}\left[w_1\left(\hat{x}_1\sqrt{m_1}\cos\frac{\theta}{2} + \hat{x}_2\sqrt{m_2}\sin\frac{\theta}{2}\right) + i\left(\left(\hat{P}_1/\sqrt{m_1}\right)\cos\frac{\theta}{2} + \left(\hat{P}_2/\sqrt{m_2}\right)\sin\frac{\theta}{2}\right)\right], \tag{90}$$

$$\hat{a}_2 = \frac{1}{\sqrt{2\hbar w_2}}\left[w_2\left(-\hat{x}_1\sqrt{m_1}\sin\frac{\theta}{2}+\hat{x}_2\sqrt{m_2}\cos\frac{\theta}{2}\right)+i\left(-\left(\hat{p}_1/\sqrt{m_1}\right)\sin\frac{\theta}{2}+\left(\hat{p}_2/\sqrt{m_2}\right)\cos\frac{\theta}{2}\right)\right]. \tag{91}$$

Before we examine their result for $\hat{I}_Z(t)$ represented in terms of $\hat{a}_i$ and $\hat{a}_i^+$, let us look into whether these ladder operators that they have chosen are no problems. The square parentheses in $\hat{a}_i$ given above are composed of position terms and momentum terms. If we consider that the dimension of w_i is T^{-1}, the dimension of position terms in it is LMT^{-1}, whereas the dimension of momentum terms is LT^{-1} (here, L, M, and T mean length, mass, and time, respectively). Hence, the two classes of terms are dimensionally not equivalent, i.e., there is a dimensional problem in this choice of $\hat{a}_i$. Moreover, the commutation relations associated to such ladder operators are given by

$$[\hat{a}_1, \hat{a}_1^+] = \frac{1}{\sqrt{m_1}}\cos^2\frac{\theta}{2}+\frac{1}{\sqrt{m_2}}\sin^2\frac{\theta}{2}, \tag{92}$$

$$[\hat{a}_2, \hat{a}_2^+] = \frac{1}{\sqrt{m_1}}\sin^2\frac{\theta}{2}+\frac{1}{\sqrt{m_2}}\cos^2\frac{\theta}{2}, \tag{93}$$

$$[\hat{a}_1, \hat{a}_2] = \frac{1}{4}\left(\sqrt{\frac{w_1}{w_2}}-\sqrt{\frac{w_2}{w_1}}\right)\left(\frac{1}{\sqrt{m_1}}-\frac{1}{\sqrt{m_2}}\right)\sin\theta, \tag{94}$$

$$[\hat{a}_1, \hat{a}_2^+] = -\frac{1}{4}\left(\sqrt{\frac{w_1}{w_2}}+\sqrt{\frac{w_2}{w_1}}\right)\left(\frac{1}{\sqrt{m_1}}-\frac{1}{\sqrt{m_2}}\right)\sin\theta, \tag{95}$$

$$[\hat{a}_1^+, \hat{a}_2] = \frac{1}{4}\left(\sqrt{\frac{w_1}{w_2}}+\sqrt{\frac{w_2}{w_1}}\right)\left(\frac{1}{\sqrt{m_1}}-\frac{1}{\sqrt{m_2}}\right)\sin\theta, \tag{96}$$

which are, at a glance, very awkward. Because $[\hat{a}_i, \hat{a}_i^+]$ are not unity (but complicated forms), the operators, $\hat{a}_i$ and $\hat{a}_i^+$ that they have defined, are not so useful in unfolding quantum theory of the given system. These operators are unity only when $m_1 = m_2 = 1$. Anyhow, they rewrote $\hat{I}_Z(t)$ in terms of $\hat{a}_i$ and $\hat{a}_i^+$ as (see Eq. 14 in Ref. (Zerimeche et al., 2022b))

$$\hat{I}_Z(t) = \sum_{i=1}^{2}\hbar w_i\left(\hat{a}_i^+\hat{a}_i+\frac{1}{2}\right). \tag{97}$$

We have checked whether this representation is right or not (see the latter part of Appendix B). We confirmed that the correct formula of $\hat{I}_Z(t)$ is not represented as Eq. (97), but it is expressed as Eq. (B12) instead. Equation (B12), which is a complicated form, is derived using its previous formula given in Eq. (B1) (that is, we did not use Zerimeche *et al*.'s wrong form given in Eq. (85)). There are two main reasons why Eq. (B12) is complicated. The first reason is that the intermediate consequence, Eq. (B4), in its derivation involves a cross term ($\hat{x}_1\hat{x}_2$ term), and the second is due to the abnormal consequences of the commutation relation $[\hat{a}_i, \hat{a}_i^+]$ shown in Eqs. (92) and (93). Even in the case where we use the Zerimeche *et al*.'s form, Eq. (85), instead of Eq. (B1), the cross term $\hat{x}_1\hat{x}_2$ does not disappear and, consequently, the outcome (ladder operators representation of $\hat{I}_Z(t)$) is not equal to Eq. (97), but complicated as ever.

Overall, we conclude that

$$\hat{I}_Z(t) \text{ in Eq. (78)} \quad \neq \quad \hat{I}_Z(t) \text{ in Eq. (85)} \quad \neq \quad \hat{I}_Z(t) \text{ in Eq. (97)}. \tag{98}$$

Equations (78), (85), and (97) correspond to Eqs. 2, 9, and 14 in Zerimeche *et al*.'s paper (Ref. (Zerimeche et al., 2022b)), respectively. We also point out that their transformation, Eqs. (83) and (84), induces intrinsic mathematical flaws for the resultant invariant operator.

We tried to remedy the transformational problem of Zerimeche *et al*. in Appendix C, by introducing improved ladder operators $\hat{B}_i$ and $\hat{b}_i$ (and their Hermitian adjoints), which are better and preferable to Zerimeche *et al*.'s original ones $\hat{a}_i$. However, as can be seen from Appendix C, the mathematical structure related to the transformed invariant operator is still incomplete and problematic despite of the use of our reformed ladder operators. Such a problem in the re-organized theory is unavoidable unfortunately, but tells us that Zerimeche *et al*.'s transformation, Eqs. (83) and (84), has an intrinsic flaw. It turned out that this flaw limits the validity of their transformation to a particular case where the coupling between the two oscillators is removed. This problem is tied to the consequence that the transformed position and momentum variables in their transformation scheme do not satisfy the standard commutation relation between them.

In summary, the problems of Zerimeche *et al*.'s transformation and their relevant evaluations in Ref. (Zerimeche et al., 2022b) are three folds. They are as follows:

i) Their evaluation of the invariant operator in each step is inconsistent as can be seen from Eq, (98). In particular, $\hat{\mathrm{I}}_Z\,(t)$ in Eq. (97), which they have obtained in the final step, is quite different from Eq. (B12) that is the representation of the invariant operator in terms of $\hat{a}_i$ and $\hat{a}_i^+$ with correction.

ii) Even in the case where the problem of i) has been fixed, there still remains a problem imbedded in their constructed ladder operators. Due to this, the corrected representation of their invariant operator, Eq. (B12), is very complicated, whereas it is not useful in unfolding the given quantum theory. $\hat{\mathrm{I}}_Z\,(t)$ in Eq. (B12) not only deviates from the standard form, but mathematically meaningless because the terms in it are dimensionally inconsistent with one another.

iii) Even when both problems pointed out in i) and ii) are fixed by choosing standard formulae of ladder operators instead of $\hat{a}_i$ and $\hat{a}_i^+$, there is another intrinsic problem regarding their transformation, which makes the transformed invariant operator abnormal and unapplicable to the system from a very general point of view. Their transformation, Eqs. (83) and (84), can be applicable only in the case where the coupling strength c_3 is zero.

In Ref. (Zerimeche et al, 2022b), they did not proceed further evaluation for the two coupled oscillators after the transformations of the invariant operator. However, we know that, if we unfold the quantum theory using mis-transformed invariant operators, the results are naturally wrong. We have previously shown that their transformed invariant operator in Ref. (Zerimeche et al., 2023) is also not right and very problematic. Moreover, recall the fact that even the original invariant operator in that paper makes no sense because η_1 and η_2 are not the same as each other (see item (1) in Subsec. 3.2.2.).

Our consequence along this line is nothing but the affirmation that Zerimeche *et al*.'s suggestion of the transformed invariant operators in Ref. (Zerimeche et al., 2022b) (together with that in Ref. (Zerimeche et al., 2023)) cannot be used in quantum theory, because they never give correct eigenvalues and eigenstates. While it is not easy to find all errors in the works of Zerimeche *et al*., we think that it is now time to put an end to represent them here.

5. CONCLUSIONS

Using a direct mathematical procedure, we have shown that the application of the invariant operator in solving the quantum problem of time-dependent coupled oscillators can be done without any approximation or perturbation manipulations. The exactness of quantum solutions in this context may enable us to understand various quantum aspects regarding coupled devices as quantum technological implements in quantum information science. In particular, the understanding of entanglement between devices are important as a core quantum effect that can be universally utilized in this realm, whereas our demonstration gives substantial benefits for the precise mathematical analysis regarding this. We also provided some perspectives and criticisms in relation with recent works in this field, which are Refs. (Choi, 2022a), (Zerimeche et al., 2023), (Choi, 2022b), etc.

Appendix A: The Full Unitary Transformation for Zerimeche *et al*.'s Work, Ref. (Zerimeche et al., 2023)

From item (2) in Subsec. 3.2.2, we have pointed out the mistake of Zerimeche *et al*. in Ref. (Zerimeche et al., 2023) regarding their choice of $\hat{U}^B_{Z,1}$. We will see in this Appendix how this mistake leads the result of their whole unitary transformation from a rigorous evaluation.

At first, they introduced $\hat{U}_{Z,1} = \hat{U}^A_{Z,1}\hat{U}^B_{Z,1}$ (see Eq. 20 in Ref. (Zerimeche et al., 2023)) and transformed the invariant operator $\hat{I}_Z$ given in Eq. (38) as

$$\hat{I}'_Z = \hat{U}_{Z,1}\,\hat{I}_Z\,U^+_{Z,1}. \tag{A1}$$

If we insert $\hat{U}^A_{Z,1}$ and $\hat{U}^B_{Z,1}$ in the above equation, which are the second and third equations in Subsec. 3.1.1 respectively, we have

$$\begin{aligned}\hat{I}'_Z &= \frac{1}{2}\sum_{i=1}^{2}\left[2\hat{p}_i^2 + \beta_i(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i) + (2\alpha_i\gamma_i - \beta_i^2)\hat{x}_i^2\right] \\ &+ \exp[i\hat{\Theta}(\hat{x}_1,\hat{x}_2)]\hat{J}_1 + \hat{J}_2\exp[i\hat{\Theta}(\hat{x}_1,\hat{x}_2)] + \hat{J}_1\exp[-i\hat{\Theta}(\hat{x}_1,\hat{x}_2)] + \exp[-i\hat{\Theta}(\hat{x}_1,\hat{x}_2)]\hat{J}_2 \\ &+ \eta\sqrt{\alpha_1\alpha_2}\,\hat{x}_1\hat{x}_2\{1+\cos[\hat{\Theta}(\hat{x}_1,\hat{x}_2)]\},\end{aligned} \tag{A2}$$

where

$$\hat{J}_i = \frac{1}{2}[\hat{p}_i^2 + \beta_i(\hat{x}_i\hat{p}_i + \hat{p}_i\hat{x}_i) + \alpha_i\gamma_i\hat{x}_i^2], \tag{A3}$$

$$\hat{\Theta}(\hat{y}_1, \hat{y}_2) = \frac{1}{2\hbar}(\beta_1 \hat{y}_1^2 - \beta_2 \hat{y}_2^2). \tag{A4}$$

Then, they further transformed $\hat{I}'_Z$ in a way that

$$\hat{I}''_Z = \hat{U}_{Z,2} \hat{I}'_Z \hat{U}^{+}_{Z,2}, \tag{A5}$$

where the formula of $\hat{U}_{Z,2}$ is given in fourth equation in Subsec. 3.1.1. Now the straightforward evaluation leads to

$$\begin{aligned}\hat{I}''_Z = {} & \frac{1}{2}\sum_{i=1}^{2}\left[2\hat{\mathrm{P}}_i^2 + \beta_i(\hat{\mathrm{X}}_i\hat{\mathrm{P}}_i + \hat{\mathrm{P}}_i\hat{\mathrm{X}}_i) + (2\alpha_i\gamma_i - \beta_i^2)\hat{\mathrm{X}}_i^2\right] \\ & + \exp[i\hat{\Theta}(\hat{\mathrm{X}}_1, \hat{\mathrm{X}}_2)]\hat{\mathrm{J}}_1 + \hat{\mathrm{J}}_2\exp[i\hat{\Theta}(\hat{\mathrm{X}}_1, \hat{\mathrm{X}}_2)] + \hat{\mathrm{J}}_1\exp[-i\hat{\Theta}(\hat{\mathrm{X}}_1, \hat{\mathrm{X}}_2)] + \exp[-i\hat{\Theta}(\hat{\mathrm{X}}_1, \hat{\mathrm{X}}_2)]\hat{\mathrm{J}}_2 \\ & + \eta\sqrt{\alpha_1\alpha_2}\,\hat{\mathrm{X}}_1\hat{\mathrm{X}}_2\{1 + \cos[\hat{\Theta}(\hat{\mathrm{X}}_1, \hat{\mathrm{X}}_2)]\},\end{aligned} \tag{A6}$$

where

$$\begin{pmatrix}\hat{\mathrm{X}}_1\\ \hat{\mathrm{X}}_2\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix}, \tag{A7}$$

$$\begin{pmatrix}\hat{\mathrm{P}}_1\\ \hat{\mathrm{P}}_2\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}, \tag{A8}$$

$$\hat{\mathrm{J}}_i = \frac{1}{2}[\hat{\mathrm{P}}_i^2 + \beta_i(\hat{\mathrm{X}}_i\hat{\mathrm{P}}_i + \hat{\mathrm{P}}_i\hat{\mathrm{X}}_i) + \alpha_i\gamma_i\hat{\mathrm{X}}_i^2]. \tag{A9}$$

This outcome is not a decoupled invariant operator and very different from the one that Zerimeche *et al*. provided in Ref. (Zerimeche et al., 2023) (see Eq. 33 in Ref. (Zerimeche et al., 2023) or Eq. (56) in this work). We see that, due to their non-negligible mistakes in adopting Choi's theory, the real content of their paper has radically gone astray.

Appendix B: The Correct Formulae of $\hat{\mathrm{I}}_Z(t)$ Given in Eqs. (85) and (97)

To reserve the validity of a quantum theory formulated based on an invariant operator, the exact development of the invariant operator in each step is indispensable. Meanwhile, it is not difficult to show that Eq. (85) is not right from a straightforward evaluation. We have re-calculated it based on the transformation represented in Eqs. (83) and (84) and found that the correct result is given by

$$\hat{\mathrm{I}}_Z(t) = \frac{1}{2}\sum_{i=1}^{2}\left(\hat{P}_i^2 + w_i^2\hat{Q}_i^2\right) + \bar{\Gamma}(t)\hat{Q}_1\hat{Q}_2, \tag{B1}$$

where

$$\begin{aligned}\bar{\Gamma}(t) = {} & \left(\frac{\dot{m}_1^2}{8m_1^2} - \frac{\dot{m}_2^2}{8m_2^2} - \frac{\int_0^t[c_1\dot{m}_1/m_1]dt}{2m_1} + \frac{\int_0^t[c_2\dot{m}_2/m_2]dt}{2m_2}\right)\sin\theta \\ & + \frac{\int_0^t c_3[\dot{m}_1/(2m_1) + \dot{m}_2/(2m_2)]dt}{2\sqrt{m_1m_2}}\cos\theta.\end{aligned} \tag{B2}$$

Now we check Eq. (97). To see whether Eq. (97) is correct or not, we consider the inverse relation of Eq. (83), which is of the form

$$\begin{pmatrix}\hat{Q}_1\\ \hat{Q}_2\end{pmatrix} = \begin{pmatrix}\sqrt{m_1(t)}\cos\frac{\theta}{2} & \sqrt{m_2(t)}\sin\frac{\theta}{2}\\ -\sqrt{m_1(t)}\sin\frac{\theta}{2} & \sqrt{m_2(t)}\cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix}. \tag{B3}$$

Then, using this relation, we can re-express Eq. (B1) under the condition in Eq. (89) as

$$\hat{\mathrm{I}}_Z(t) = \frac{1}{2}\sum_{i=1}^{2}\left(\hat{P}_i^2 + W_i^2\hat{x}_i^2\right) + \tilde{\Gamma}(t)\hat{x}_1\hat{x}_2, \tag{B4}$$

where

$$W_1^2(t) = m_1\left(w_1^2\cos^2\frac{\theta}{2} + w_2^2\sin^2\frac{\theta}{2}\right), \tag{B5}$$

$$W_2^2(t) = m_2\left(w_1^2\sin^2\frac{\theta}{2} + w_2^2\cos^2\frac{\theta}{2}\right), \tag{B6}$$

$$\tilde{\Gamma}(t) = \sqrt{m_1 m_2}(w_1^2 - w_2^2)\cos\frac{\theta}{2}\sin\frac{\theta}{2}. \tag{B7}$$

To proceed further, we regard the inverse relations of Eqs. (90) and (91), which are

$$\hat{x}_1 = \sqrt{\frac{\hbar}{2m_1}}\left(\frac{\cos(\theta/2)}{\sqrt{W_1}}(\hat{a}_1 + \hat{a}_1^+) - \frac{\sin(\theta/2)}{\sqrt{W_2}}(\hat{a}_2 + \hat{a}_2^+)\right), \tag{B8}$$

$$\hat{x}_2 = \sqrt{\frac{\hbar}{2m_2}}\left(\frac{\sin(\theta/2)}{\sqrt{W_1}}(\hat{a}_1 + \hat{a}_1^+) + \frac{\cos(\theta/2)}{\sqrt{W_2}}(\hat{a}_2 + \hat{a}_2^+)\right), \tag{B9}$$

$$\hat{P}_1 = i\sqrt{\frac{\hbar m_1}{2}}\left(\sqrt{W_1}(\hat{a}_1^+ - \hat{a}_1)\cos\frac{\theta}{2} - \sqrt{W_2}(\hat{a}_2^+ - \hat{a}_2)\sin\frac{\theta}{2}\right), \tag{B10}$$

$$\hat{P}_2 = i\sqrt{\frac{\hbar m_2}{2}}\left(\sqrt{W_1}(\hat{a}_1^+ - \hat{a}_1)\sin\frac{\theta}{2} + \sqrt{W_2}(\hat{a}_2^+ - \hat{a}_2)\cos\frac{\theta}{2}\right). \tag{B11}$$

Using Eqs. (B8)-(B11), it is confirmed that Eq. (B4) becomes

$$\hat{\mathrm{I}}_Z(t) = \hbar\sum_{i=1}^{2}\left[\mu_i(t)\hat{a}_i^+\hat{a}_i - \nu_i(t)(\hat{a}_i^2 + \hat{a}_i^{+2})\right] + \hbar\varsigma(t)(\hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_2^+ - \hat{a}_1^+\hat{a}_2 + \hat{a}_1^+\hat{a}_2^+) + \hbar\chi(t), \tag{B12}$$

where

$$\mu_1(t) = \frac{1}{4}W_1[m_1 + m_2 + (m_1 - m_2)\cos\theta + 2], \tag{B13}$$

$$\mu_2(t) = \frac{1}{4}W_2[m_1 + m_2 - (m_1 - m_2)\cos\theta + 2], \tag{B14}$$

$$\nu_1(t) = \frac{1}{8}W_1[m_1 + m_2 + (m_1 - m_2)\cos\theta - 2], \tag{B15}$$

$$\nu_2(t) = \frac{1}{8}W_2[m_1 + m_2 - (m_1 - m_2)\cos\theta - 2], \tag{B16}$$

$$\varsigma(t) = \frac{1}{4}\sqrt{W_1W_2}(m_1 - m_2)\sin\theta, \tag{B17}$$

$$\chi(t) = \frac{W_1 - W_2}{8}\left(\frac{1}{\sqrt{m_1}} - \frac{1}{\sqrt{m_2}} + \sqrt{m_1} - \sqrt{m_2}\right)\cos\theta + \frac{W_1 + W_2}{32}\left(\frac{1}{\sqrt{m_1}} + \frac{1}{\sqrt{m_2}}\right) \times\left[4 + \left(\sqrt{m_1} + \sqrt{m_2}\right)^2 - \left(\sqrt{m_1} - \sqrt{m_2}\right)^2\cos(2\theta)\right]. \tag{B18}$$

Curiously, $\mu_i(t)$, $\nu_i(t)$, and $\chi(t)$ are not only complicated forms, but they also involve mathematically unreasonable expressions, such as "$1/\sqrt{m_1} - 1/\sqrt{m_2} + \sqrt{m_1} - \sqrt{m_2}$" (see $\chi(t)$). That is, $1/\sqrt{m_i}$ and $\sqrt{m_i}$ dimensionally do not match each other, while they are mutually added and subtracted in this expression. This problem originates from Zerimeche *et al.*'s problematic choice of dimensions in $\hat{a}_i$ and $\hat{a}_i^+$, which was mentioned in the text.

Appendix C: Exploring Intrinsic Flaws in Zerimeche *et al.*'s Transformation in Ref. (Zerimeche et al., 2022b)

We have shown in Appendix B that Zerimeche *et al.*'s formulation for the ladder operators in Ref. (Zerimeche et al., 2022b) is problematic. The expression given in Eq. (B12) is unfamiliar and very complicated. Here, we show that, in addition to it, there is also an intrinsic drawback in their transformation, by showing that there remains a problem even when the ladder-operator problem is fixed.

C1. Basic search with the choice of standard ladder operators

For the purposes mentioned above, let us see whether we can improve the formula of the transformed invariant operator by choosing ladder operators alternatively. Through such an examination for the improvement, we explore what is the problem of the Zerimeche *et al.*'s transformation. We present a more plausible ladder operator representation of $\hat{I}_Z(t)$ in connection with Zerimeche *et al.*'s particular transformation given in Eqs. (83) and (84). The standard ladder operators (annihilation operators) that we consider as alternations to Zerimeche *et al.*'s ones are of the forms

$$\hat{B}_i = \sqrt{\frac{w_i}{2\hbar}}\hat{Q}_i + \frac{i}{\sqrt{2w_i\hbar}}\hat{P}_i, \tag{C1}$$

$$\hat{b}_1 = \sqrt{\frac{w_1}{2\hbar}}\left(\hat{x}_1\sqrt{m_1}\cos\frac{\theta}{2} + \hat{x}_2\sqrt{m_2}\sin\frac{\theta}{2}\right) + \frac{i}{\sqrt{2m_1w_1\hbar}}\left(\hat{p}_1 + \frac{\dot{m}_1}{2}\hat{x}_1\right), \tag{C2}$$

$$\hat{b}_2 = \sqrt{\frac{w_2}{2\hbar}}\left(-\hat{x}_1\sqrt{m_1}\sin\frac{\theta}{2} + \hat{x}_2\sqrt{m_2}\cos\frac{\theta}{2}\right) + \frac{i}{\sqrt{2m_2w_2\hbar}}\left(\hat{p}_2 + \frac{\dot{m}_2}{2}\hat{x}_2\right), \tag{C3}$$

while their Hermitian adjoints, $\hat{B}_i^+$ and $\hat{b}_i^+$, are creation operators. $\hat{B}_i$ are well-known annihilation operators associated with Eq. (B1) under the condition that the last term in it is removed by choosing θ as Eq. (89). We assumed that the transformed systems associated with $\hat{I}_Z(t)$ in Eq. (B1) are uncoupled ideal SHOs in defining the formulae of $\hat{B}_i$. That is, anomalies, time-dependencies, etc. of $\hat{I}_Z(t)$ are not considered in formulating them, if there are. $\hat{b}_i$, on the other hand, are $(\hat{x}_i, \hat{p}_i)$-spaces representations of $\hat{B}_i$. These operators satisfy the commutation relations

$$\left[\hat{B}_i, \hat{B}_i^+\right] = \left[\hat{b}_i, \hat{b}_i^+\right] = \cos\frac{\theta}{2}, \tag{C4}$$

which deviate from unity depending on θ. The inverse representations of Eqs. (C2) and (C3) together with their Hermitian adjoints are given by

$$\hat{x}_1 = \sqrt{\frac{\hbar}{2m_1w_1w_2}}\left(\sqrt{w_2}\cos\frac{\theta}{2}(\hat{b}_1 + \hat{b}_1^+) - \sqrt{w_1}\sin\frac{\theta}{2}(\hat{b}_2 + \hat{b}_2^+)\right), \tag{C5}$$

$$\hat{x}_2 = \sqrt{\frac{\hbar}{2m_2w_1w_2}}\left(\sqrt{w_2}\sin\frac{\theta}{2}(\hat{b}_1 + \hat{b}_1^+) + \sqrt{w_1}\cos\frac{\theta}{2}(\hat{b}_2 + \hat{b}_2^+)\right), \tag{C6}$$

$$\hat{p}_1 = \sqrt{\frac{\hbar}{2m_1w_1w_2}}\left\{\sqrt{w_2}\left[\left(-\frac{\dot{m}_1}{2}\cos\frac{\theta}{2} - im_1w_1\right)\hat{b}_1 + \left(-\frac{\dot{m}_1}{2}\cos\frac{\theta}{2} + im_1w_1\right)\hat{b}_1^+\right] + \frac{\dot{m}_1}{2}\sqrt{w_1}\sin\frac{\theta}{2}(\hat{b}_2 + \hat{b}_2^+)\right\}, \tag{C7}$$

$$\hat{p}_2 = \sqrt{\frac{\hbar}{2m_2w_1w_2}}\left\{\sqrt{w_1}\left[\left(-\frac{\dot{m}_2}{2}\cos\frac{\theta}{2} - im_2w_2\right)\hat{b}_2 + \left(-\frac{\dot{m}_2}{2}\cos\frac{\theta}{2} + im_2w_2\right)\hat{b}_2^+\right] - \frac{\dot{m}_2}{2}\sqrt{w_2}\sin\frac{\theta}{2}(\hat{b}_1 + \hat{b}_1^+)\right\}. \tag{C8}$$

Then, under the condition in Eq. (89), $\hat{I}_Z(t)$ given in Eq. (B1) can be expressed in a simple form as

$$\hat{I}_Z(t) = \sum_{i=1}^{2}\hbar w_i\left(\hat{B}_i^+\hat{B}_i + \frac{1}{2}\cos\frac{\theta}{2}\right). \tag{C9}$$

These representations are much more brief and to the point than the corrected representation of Zerimeche *et al.*'s formula given in Eq. (B12). We, however, see an anomaly in these expressions, which is that the zero-point eigenvalue of $\hat{I}_Z(t)$ is not $(1/2)\Sigma_{i=1}^{2}\hbar w_i$, but $(1/2)\Sigma_{i=1}^{2}\hbar w_i\cos(\theta/2)$. Moreover, it is unable to say that $\hat{B}_i^+\hat{B}_i$ are (normal) number operators because the allowed eigenvalues of them are not $n_i = 0, 1, 2, \cdots$, but

$$n_i = 0, \cos\frac{\theta}{2}, 2\cos\frac{\theta}{2}, \cdots. \tag{C10}$$

It may also be worthwhile to check the commutation relations between $\hat{B}_1$ and $\hat{B}_2$ including their Hermitian adjoints. They are as follows:

$$\left[\hat{B}_1, \hat{B}_2\right] = \left[\hat{b}_1, \hat{b}_2\right] = -\frac{1}{2}\left(\sqrt{\frac{w_1}{w_2}} + \sqrt{\frac{w_2}{w_1}}\right)\sin\frac{\theta}{2}, \tag{C11}$$

$$\left[\hat{B}_1, \hat{B}_2^+\right] = \left[\hat{b}_1, \hat{b}_2^+\right] = \frac{1}{2}\left(\sqrt{\frac{w_1}{w_2}} - \sqrt{\frac{w_2}{w_1}}\right)\sin\frac{\theta}{2}, \tag{C12}$$

$$\left[\hat{B}_1^+, \hat{B}_2\right] = \left[\hat{b}_1^+, \hat{b}_2\right] = -\frac{1}{2}\left(\sqrt{\frac{w_1}{w_2}} - \sqrt{\frac{w_2}{w_1}}\right)\sin\frac{\theta}{2}. \tag{C13}$$

These relations are not zero and much more abnormal than those in Eq. (C4). We can also easily represent $\hat{I}_Z(t)$ in terms of $\hat{b}_i$ and $\hat{b}_i^+$ by inserting Eqs. (C5)-(C8) into Eq. (78), but its expression is very complicated on account of the anomaly in the associated mathematical structures.

C2. Analysis from a different angle by re-formulating operators

Because the commutation relations in Eq. (C4) are not unity, let us seek another possibility which makes it unity. For that purpose, we introduce slight different definition of ladder operators such that

$$\hat{\mathrm{B}}_i = \frac{\hat{B}_i}{\sqrt{\cos(\theta/2)}}, \tag{C14}$$

$$\hat{\mathrm{b}}_i = \frac{\hat{b}_i}{\sqrt{\cos(\theta/2)}}. \tag{C15}$$

Then, instead of Eq. (C4), we have

$$\left[\hat{\mathrm{B}}_i, \hat{\mathrm{B}}_i^+\right] = \left[\hat{\mathrm{b}}_i, \hat{\mathrm{b}}_i^+\right] = 1, \tag{C16}$$

which are unity. We also have the invariant operator in terms of $\hat{\mathrm{B}}_i$ and $\hat{\mathrm{B}}_i^+$ as

$$\hat{I}_Z(t) = \sum_{i=1}^{2} \mathrm{h} w_i\left(\hat{\mathrm{B}}_i^+\hat{\mathrm{B}}_i + \frac{1}{2}\right), \tag{C17}$$

where

$$\mathrm{h} = \hbar\cos\frac{\theta}{2}. \tag{C18}$$

Though it seems, superficially, that the problem inherent in Eq. (C9) is fixed in the case of Eq. (C17), it is actually not fixed. Because of the relation in Eq. (C18), Eq. (C17) is in fact the same as Eq. (C9).

Moreover, in spite of the improvement of the basic commutation relations as can be seen from Eq. (C16), the remnant commutation relations are still complicated:

$$\left[\hat{\mathrm{B}}_1, \hat{\mathrm{B}}_2\right] = \left[\hat{\mathrm{b}}_1, \hat{\mathrm{b}}_2\right] = -\frac{1}{2}\left(\sqrt{\frac{w_1}{w_2}} + \sqrt{\frac{w_2}{w_1}}\right)\tan\frac{\theta}{2}, \tag{C19}$$

$$\left[\hat{\mathrm{B}}_1, \hat{\mathrm{B}}_2^+\right] = \left[\hat{\mathrm{b}}_1, \hat{\mathrm{b}}_2^+\right] = \frac{1}{2}\left(\sqrt{\frac{w_1}{w_2}} - \sqrt{\frac{w_2}{w_1}}\right)\tan\frac{\theta}{2}, \tag{C20}$$

$$\left[\hat{\mathrm{B}}_1^+, \hat{\mathrm{B}}_2\right] = \left[\hat{\mathrm{b}}_1^+, \hat{\mathrm{b}}_2\right] = -\frac{1}{2}\left(\sqrt{\frac{w_1}{w_2}} - \sqrt{\frac{w_2}{w_1}}\right)\tan\frac{\theta}{2}. \tag{C21}$$

Hence, the problem of the mathematical structures imbedded in the operators representation cannot be fixed by introducing different ladder operators. This means that the transformation in Eqs. (83) and (84) is accompanied by intrinsic mathematical flaws that cannot be remedied by other means unless we rely on a certified formula of transformations instead of it.

C3. For the Case of $\theta = 0$

Finally, we consider a specific case where $\theta = 0$ in this context. Anomalies between ladder operators, which we have previously seen, disappear in this case. Accordingly, both expressions of the invariant operators, Eqs. (C9) and (C17), reduce to a familiar one:

$$\hat{I}_Z(t) = \sum_{i=1}^{2} \hbar w_i\left(\hat{B}_i^+\hat{B}_i + \frac{1}{2}\right), \tag{C22}$$

where we considered the relations $\hat{\mathrm{B}}_i = \hat{B}_i$ that hold in this particular case. The expression in Eq. (C22) is no problem. In this case, the invariant operator can also be represented in a simple form even in terms of $\hat{b}_i$ and $\hat{b}_i^+$ such that

$$\hat{\mathrm{I}}_Z(t) = \sum_{i=1}^{2} \hbar \overline{w}_i \left(\hat{b}_i^+ \hat{b}_i + \frac{1}{2} \right), \tag{C23}$$

where

$$\overline{w}_i = w_i \big|_{\theta=0} = \sqrt{\alpha_i \gamma_i - \beta_i^2}\,. \tag{C24}$$

We have seen that, though the corrected formula of $\hat{\mathrm{I}}_Z(t)$ in Zerimeche *et al.*'s development, given in Eq. (B12), is reduced to a simple one if we represent it in terms of $\hat{B}_i$ and $\hat{B}_i^+$ instead of $\hat{a}_i$ and $\hat{a}_i^+$, it is still problematic. The problem can be fixed only when $\theta = 0$. Because $\theta = 0$ corresponds to $c_3 = 0$, Zerimeche *et al.*'s transformation, Eqs. (83) and (84), can be applied only to the non-coupled systems of which quantum solutions are already well-known. This consequence stems from the fact that the commutation relations between the transformed canonical variables are non-standard, namely, they are $[\hat{Q}_i, \hat{P}_i] \neq i\hbar$, where this anomaly can be easily verified on the basis of the basic conditions $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. The abnormal consequences in Eqs. (92)-(96) are also partly related to this.

Author's notes

1. Some mathematical notations in this work are not the same as the ones in the cited papers, such as Refs. (Choi, 2022a; Choi, 2022b; Zerimeche et al., 2023; Zerimeche et al., 2022a; Zerimeche et al., 2022b). These differences are for the purpose of keeping our own consistency in the representation.
2. The rules applied on citations:
(1) We have cited equations in this work as Eq. (X), whereas we have cited equations appeared in other works as Eq. X for the facility of distinction.
(2) We have cited sections and subsections in this work as Sec. X and Subsec. X, whereas we have cited those appeared in other works as section X and subsection X.
(3) A reference in this work has been written as Ref. (Author(s), Year) in the text, and a reference appeared in other works as Ref. X.

Authors contributions

JNS wrote the paper and approved it.

Ethical approval

Not applicable.

Informed consent

Not applicable.

Conflicts of interests

The authors declare that there are no conflicts of interests.

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Data and materials availability

All data associated with this study are present in the paper.

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